MATH 3100 – Homework #6
posted October 25, 2024; due November 4, 2024

Section and exercise numbers correspond to the online notes. Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.**

Required problems

1. §2.3: 11.

2. Suppose $\sum_{k=0}^{\infty} a_k x^k$ is a power series whose DOC has least upper bound R, where $0 < R < \infty$. Show that $\sum_{k=0}^{\infty} a_k x^k$ converges when $|x| < R$ and diverges when $|x| > R$.

   This completes the proof of the theorem from class characterizing the possible forms of the DOC. Needless to say, you should not assume that theorem in your proof!

3. §2.4: 1(b,e,f,k)

4. (a) Let f be a real-valued function with domain D and let $x_0 \in D$. The statement that f is continuous at x_0 can be written, in logical notation, as

   $$(\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x \in D)\ (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon).$$

   Write out the **negation** of this statement, again in logical notation.

   (b) Let f be the function defined by $f(x) = 0$ when $x \leq 0$ and $f(x) = 1$ when $x > 0$. Using (a), show that f is not continuous at $x_0 = 0$. Do *not* use the sequential criterion for continuity.

5. Let f be a real-valued function with domain D and let $x_0 \in D$. In class, we stated that continuity at x_0 and sequential continuity at x_0 are equivalent.

   We proved one direction: Continuity at x_0 implies sequential continuity at x_0. In this exercise you are asked to tackle the other direction. That is, supposing that f is not continuous at x_0, prove that f is not sequentially continuous at x_0.

   (a) Suppose f is not continuous at x_0. Show that there is some $\epsilon > 0$ with the following property: For every $n \in \mathbf{N}$, there is an $x_n \in D$ with $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \epsilon$.

   (b) Show that if the x_n are as in part (a), then $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$. (Hence, f is not sequentially continuous at x_0.)

6. (a) Show that for every pair of real numbers x and y,

   $$-|x - y| \leq |x| - |y| \leq |x - y|.$$

   Hint. Use the triangle inequality, writing $y = x + h$ for some h.

   (b) Show that $|x|$ is continuous. (We took this for granted earlier in the semester!)

7. Assume that the function "$\sqrt{x}$" makes sense. That is, there is a real-valued function labeled $\sqrt{x}$ defined for all $x \geq 0$ satisfying

(a) $\sqrt{x} \geq 0 \quad$ for all $x \geq 0$,

(b) $(\sqrt{x})^2 = x \quad$ for all $x \geq 0$.

(This follows from the Intermediate Value Theorem, to be treated shortly!)

Prove that $\sqrt{x}$ is continuous.

Suggestion. You may wish to prove continuity at $x_0 = 0$ separately from continuity at $x_0 > 0$.

Recommended problems

§2.3: 3(a,d,e), 7, 10
Ross: 17.5, 17.6