MATH 3100 – Homework #7

posted November 13, 2024; due November 20, 2024

Limits, like fears, are often just an illusion. – Michael Jordan

Required problems

1. For this problem, you may assume that $\sin x$ and $\cos x$ are defined and continuous on all of $\mathbf{R}$. You may also assume that the function x^π, with domain $(0, \infty)$, is continuous on $(0, \infty)$. Let
$$f(x) = (\sin^2 x + \cos^6 x)^\pi.$$
   (a) What is the domain of $f(x)$? Justify your answer.
   (b) Using theorems stated in class or in the textbook (Ross), explain why $f(x)$ is continuous.

2. Let $f(x) = 2x \cos(1/x)$ for $x \neq 0$ and set $f(0) = 0$. Show that f is continuous at 0 by directly verifying the ϵ-δ definition of continuity. Do *not* use the sequential criterion.

3. Prove that $f(x) = x^3$ is continuous at $x = 2$ by directly verifying the ϵ-δ definition of continuity. Do *not* use the sequential criterion.

4. (Squeeze Lemma for sequences) Suppose $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are three sequences satisfying
$$a_n \leq b_n \leq c_n \quad \text{for all } n \in \mathbf{N}.$$
   Suppose there is a real number L such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Prove that $\lim_{n\to\infty} b_n = L$.

   Remark. This problem could have been assigned in Chapter 1; no new theory is required.

5. (Squeeze Lemma for functions) Let f_1, f_2, and f_3 be three functions defined on all of $\mathbf{R}$, and let a be a real number. Suppose that
$$f_1(x) \leq f_2(x) \leq f_3(x) \quad \text{for all } x \in \mathbf{R}.$$
   Suppose also that there is a real number L for which
$$\lim_{x\to a} f_1(x) = \lim_{x\to a} f_3(x) = L.$$
   Prove that
$$\lim_{x\to a} f_2(x) = L.$$

6. Let f be a real-valued function with domain D, and let $a \in \mathbf{R}$. Suppose that δ_1, δ_2 are positive real numbers for which $S_1 = (a - \delta_1, a + \delta_1) \setminus \{a\}$ and $S_2 = (a - \delta_2, a + \delta_2) \setminus \{a\}$ are both contained in D.

   Let $L \in \mathbf{R}$. Suppose that whenever $\{x_n\}$ is a sequence in S_1 for which $x_n \to a$, we have $f(x_n) \to L$. Prove that whenever $\{x_n\}$ is a sequence in S_2 for which $x_n \to a$, we have $f(x_n) \to L$.

   This shows that — at least in the case when $L \in \mathbf{R}$ — which δ we use to check if $\lim_{x\to a} f(x) = L$ doesn't matter, as long as $(a - \delta, a + \delta) \setminus \{a\} \subseteq D$.

Recommended problems (from Ross)

§17: 3(a,c,f,g,h), 8, 11, 15
§20: 1, 9, 11, 13