MATH 3100 – Learning objectives to meet for Exam #3

The exam will cover §2.4 of the course notes, as well as the material on continuity, limits and derivatives treated by the end of class on Monday, 11/17.

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- power series
- domain of convergence and radius of convergence
- continuity of a function f at x_0
- f is continuous (with no point specified)
- $\lim_{x\to a} f(x)$
- f is differentiable at a

Big theorems

Be familiar with all of the following theorems, making sure that you can give full statements for those not already spelled out below. For results proved in class, describe the components and main ideas of the proof.

- Key Lemma: If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$, it converges absolutely for all x with $|x| < |x_0|$
- Characterization of the domain of convergence of a power series (Proposition 2.4.4)
- Sequential criterion for continuity
- Constant multiples, sums, products, and ratios of continuous functions are continuous
- The composition of continuous functions is continuous
- Intermediate Value Theorem
- Maximum Value Theorem
- Limit laws for sums, products, and quotients
- Differentiability at a implies continuity at a
- Derivatives of constant multiples, sums, products, and quotients
- Chain rule
- Rolle's theorem
- Mean Value Theorem
- If f has a maximum at $x_0 \in (a, b)$, and f is differentiable at x_0, then $f'(x_0) = 0$. And similarly for minimum.

What to expect on the exam

There will be five questions on the exam, possibly having multiple parts. These will include...

- A problem or problem part asking you to determine the domain and/or radius of convergence of (one or more) power series
- A problem asking you to prove or disprove continuity of a given function directly from the ϵ-δ definition
- A problem requiring you to apply the Intermediate Value Theorem and/or the Maximum Value Theorem

Practice problems

1. Find the domain of convergence of $\sum_{n=0}^{\infty} nx^n$. Do the same for $\sum_{n=1}^{\infty} \frac{(-2)^n}{n} x^{3n}$. Justify your answers.

2. (a) What does it mean to say that $f : D \to \mathbf{R}$ is continuous at $x = a$? Make sure to state all of your assumptions.

   (b) Prove, directly from the ϵ-δ definition, that $f(x) = 1/x$ is continuous at $x = 1$. You may **not** use the sequential criterion for continuity or the continuity rules proved in class.

3. (Limits at infinity) Suppose $f\colon D \to \mathbf{R}$, and let $L \in \mathbf{R} \cup \{\pm\infty\}$. We say

$$\lim_{x\to\infty} f(x) = L$$

   if the following conditions hold:

   (i) $S = (z, \infty) \subseteq D$ for some real number z,

   (ii) whenever $\{x_n\}$ is a sequence in S for which $x_n \to \infty$, we have $f(x_n) \to L$.

   (a) If $f\colon D \to \mathbf{R}$ and $g\colon E \to \mathbf{R}$, what do we mean by the function $f + g$? Give me both the domain of $f + g$ and the values of $(f + g)(x)$ for each x in the domain.

   (a) Give a careful proof of the sum rule for limits at infinity. In other words, prove that if $f\colon D \to \mathbf{R}$ and $g\colon E \to \mathbf{R}$, and

$$\lim_{x\to\infty} f(x) = L, \quad \text{while} \quad \lim_{x\to\infty} g(x) = M,$$

   where L and M are real numbers, then

$$\lim_{x\to\infty} (f + g)(x) = L + M.$$

4. (a) What does it mean to say that $f(x)$ is differentiable at $x = a$?

   (b) State the Mean Value Theorem.

(c) Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable on all of $\mathbf{R}$ and that $f'(x) \leq 2$ for all $x \in \mathbf{R}$. If $f(0) = 0$, what is the largest possible value of $f(100)$? Justify your answer. Make sure you explain why your "largest possible" value is actually possible.

5. Let f be a continuous function on $[0, 1]$. Suppose that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$, with $f(0) > 0$ and $f(1) < 1$. Use the Intermediate Value Theorem to prove that the graph of $y = f(x)$ on the interval $[0, 1]$ intersects the graph of $y = x$ on $[0, 1]$.