MATH 3100/3100H homework – Homework #1
posted August 18, 2025; due August 27 by end-of-day

It requires a very unusual mind to undertake the analysis of the obvious. – A.N. Whitehead

Section and exercise numbers correspond to the notes of Dr. Adams. Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.**

1. §1.1: Exercise 4.
2. §1.1: Exercise 5.
3. §1.1: Exercise 8.
4. §1.2: Exercise 5.
5. §1.2: Exercise 10.
6. §1.2: Exercise 14(b).
7. The following is a statement of complete induction with a different base case:

    Suppose $S \subseteq \mathbf{N}$. Let $n_0 \in \mathbf{N}$, and suppose both of the following hold:

    (i) $n_0 \in S$,

    (ii) if n is a natural number with $n \geq n_0$, and all of $n_0, n_0 + 1, \ldots, n \in S$, then $n + 1 \in S$.

    Then $S \supseteq \{n \in \mathbf{N} : n \geq n_0\}$.

    We will take this as a basic principle of mathematical reasoning.

    On Wednesday, 8/20, we will discuss the following statement: *For every $n \in \mathbf{N}$ with $n \geq 12$, one can make n cents postage out of 4 cent and 5 cent stamps.* What follows below is one way of formalizing the informal argument we will give in class. Your job: **Write out the complete argument on your own sheet of paper, filling in the details**!

    Let $S = \{n \in \mathbf{N} : \text{one can make } n \text{ cents postage out of 4 and 5 cent stamps}\}$. We want to show that $S \supseteq \{n \in \mathbf{N} : n \geq 12\}$. We apply complete induction with base case $n_0 = 12$.

    First, $12 \in S$, since [fill this in!].

    Now let $n \in \mathbf{N}$ where $n \geq 12$, and assume that all of $12, 13, \ldots n \in S$. We will show $n + 1 \in S$. If $n = 12$, 13, or 14, then $n + 1 \in S$ since [fill this in !].

    Thus, we can assume $n \geq 15$. Then $n + 1 \geq 16$, and $(n + 1) - 4 \geq 12$. Therefore, [fill this in!].

    Hence, $n + 1 \in S$. By complete induction, S contains all natural numbers $n \geq 12$, as desired.

8. §1.2: Exercise 19.
9. Define real numbers α and β by $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

(a) Check that α and β are roots of the polynomial $x^2 - x - 1$.

(b) Using (a), deduce that $\alpha^{n+1} = \alpha^n + \alpha^{n-1}$ and $\beta^{n+1} = \beta^n + \beta^{n-1}$, for every integer n. (First use (a) to explain why this holds when $n = 1$. Then deduce the general case. For the general case you don't need induction, just algebra!)

(c) Recall that the Fibonacci sequence $\{F_n\}$ is defined by $F_1 = 1$, $F_2 = 1$, and the recurrence $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.
Use complete induction to prove that $\frac{\alpha^n - \beta^n}{\sqrt{5}} = F_n$ for all natural numbers n.
Hint: The result of (b) will be useful.

10. The following argument is an *alleged* proof that in any finite group of people, all of them have the same height:

> Let S be the set of natural numbers n for which the statement "every group of n people share the same height" is true. Obviously the statement is true if there is just one person, so $1 \in S$. Now we suppose that $n \in S$, and we prove that $n + 1 \in S$. Take any group of $n + 1$ people, say $A_1, \ldots, A_{n+1}$. Since $n \in S$, it must be that $A_1, \ldots, A_n$ all share the same height, and similarly for $A_2, \ldots, A_{n+1}$. But these two groups overlap; for instance, the second person A_2 is in both. So all of our $n + 1$ people have the same height (in fact, everyone is the same height as A_2). Thus, $n + 1 \in S$. So by induction, S is all of the natural numbers.

Clearly explain the mistake in the proof.

Extra problems for 3100H

These problems are mandatory for students enrolled in MATH 3100H. For 3100 students, a 3100H problem may be worked for extra credit (up to half the point value).

11. Throughout this problem, $\sqrt{2}$ denotes the positive real square root of 2, so that $\sqrt{2} = 1.414\ldots$.

    (a) Suppose that m is an integer for which $m\sqrt{2}$ is also an integer. Write $m\sqrt{2} = n$, where $n \in \mathbf{Z}$. (Remember that the symbol $\mathbf{Z}$ denotes the set of integers.) Explain why $(n - m)\sqrt{2}$ is also an integer.

    (b) Using strong induction and your observation in (a), show that there is no positive integer m for which $m\sqrt{2}$ is an integer.

    (c) What you proved in part (b) is another way of stating a famous classical theorem. Which theorem?

12. Let F_n be the nth Fibonacci number, as defined earlier in this problem set. Prove that for all natural numbers $n \geq 2$, we have $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$. You may **not** use the formula for the Fibonacci numbers derived in Problem 9. Instead, work directly from the recurrence relation defining the Fibonacci numbers.