MATH 3100 – Homework #2

posted August 29, 2025; due by end-of-day on September 8, 2025

'Obvious' is the most dangerous word in mathematics. — E.T. Bell

Section and exercise numbers correspond to the online notes. Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.**

Required problems for MATH 3100

1. Let $a_n = 2n$ for all $n \in \mathbf{N}$, and let $b_n = 2^n$ for all $n \in \mathbf{N}$. Show that $\{b_n\}$ is a subsequence of $\{a_n\}$ by writing down a strictly increasing function $g\colon \mathbf{N} \to \mathbf{N}$ with $b_n = a_{g(n)}$ for all $n \in \mathbf{N}$.
2. §1.3, Exercise 8 (no proofs necessary)
3. §1.3, Exercise 9 (no proofs necessary)
4. §1.3, Exercise 13
5. §1.3, Exercise 15
6. Let $\{a_n\}$ be the sequence defined by $a_n = n^2 - 7n + 11$ for all $n \in \mathbf{N}$. Find a value of $N \in \mathbf{N}$ such that $a_n \leq a_{n+1}$ for all $n \geq N$. Justify your answer.
7. Show that if $\{a_n\}$ is a sequence, then $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$. (Remember that an if-and-only-if statement requires a proof for **both** directions.)
8. In class, we will show that if $0 < r < 1$, then $\lim_{n\to\infty} r^n = 0$. We will also show that if $r > 1$, then $\{r^n\}$ is not bounded. You may assume these two results for this exercise.
   (a) Suppose that $-1 < r < 0$. Prove that $\lim_{n\to\infty} r^n = 0$.
   (b) Suppose that $r < -1$. Prove that $\{r^n\}$ is unbounded. Deduce that $\{r^n\}$ diverges in this case.

   Hint: Problems 5 and 7 (above) may be useful.
9. §1.4, Exercise 2
10. §1.4, Exercise 8

Recommended problems (NOT to turn in)

§1.3: 14, 17, 18, 19, 20, 25
§1.4: 1, 3, 4, 5, 6, 7

Extra problems for MATH 3100H

Both of the following following problems concern the sequence $\{x_n\}$ introduced in class to motivate the definition of convergence. Recall that $\{x_n\}$ is defined recursively by $x_1 = 2$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \quad \text{for all } n \in \mathbf{N}.$$

11. Use induction to prove that $1 \leq x_n \leq 2$ for all $n \in \mathbf{N}$.

    You may **not** use calculus: Rely only basic facts about inequalities. As an example, at some point you will want to use that if x is a real number with $x \geq 2$, then $2/x \leq 1$.

12. (a) Prove that if t is a real number with $t > 0$, then $t + \frac{2}{t} \geq 2\sqrt{2}$.

    You may **not** use calculus. Instead, use that squares of real numbers are always nonnegative.

    (b) Using (a) and the definition of the sequence $\{x_n\}$, prove that each $x_n \geq \sqrt{2}$.

    Warning: This is **not** a proof by induction — it's easier than that!

    (c) Prove that $\{x_n\}$ is a decreasing sequence. (Part (b) should be helpful here.)