MATH 3100 – Homework #3

posted September 8, 2025; due by end of day on Wednesday, September 17

When I was a child, the Earth was said to be two billion years old. Now scientists say it's four and a half billion. So that makes me two and a half billion. — Paul Erdős

Section and exercise numbers correspond to the online notes. Assignments are expected to be **neat** and **stapled**, with problems arranged in the order they appear below. **Illegible work may not be marked.**

Required problems

1. §1.4: 10
2. §1.4: 15
3. §1.4: 17
4. §1.4: 23
5. §1.5: 3
6. §1.5: 6

Recommended problems (NOT to turn in)

§1.4: 11, 20, 21, 22
§1.5: 7(a)

MATH 3100H problems

7. (Cesaro Means) Let $\{x_n\}$ be a sequence that converges to 0. Define a new sequence $\{y_n\}$ by the rule
$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad \text{for each } n \in \mathbf{N}.$$
You should think of y_n as the "running average" of the first n terms from the original sequence.

   Since convergent sequences are bounded, we can pick a real number $M \geq 0$ with each $|x_n| \leq M$.

   (a) Let $\epsilon > 0$, and choose $N_0 \in \mathbf{N}$ with $|x_n| < \frac{1}{2}\epsilon$ for all $n \geq N_0$. Prove that for all natural numbers $n \geq N_0$, we have
$$|y_n| \leq M\frac{N_0}{n} + \frac{1}{2}\epsilon.$$

   (b) Using the result of (a), prove that $y_n \to 0$.

8. (continuation)

   (a) Now suppose $\{x_n\}$ is a sequence that converges to L, where L is an arbitrary real number (not necessarily 0 as in the last problem). As before, define $\{y_n\}$ by letting
$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad \text{for each } n \in \mathbf{N}. \qquad (*)$$
Prove that $y_n \to L$.

   Don't reinvent the wheel. Find a way to use the result of Problem 7.

   (b) Explicitly describe a sequence of real numbers $\{x_n\}$ for which $\{x_n\}$ diverges but nevertheless $\{y_n\}$, as defined by (*), converges. Justify your answer. That is, prove both that $\{x_n\}$ diverges and that the corresponding sequence $\{y_n\}$ converges.