MATH 3100 – Homework #4

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Answer the questions, then question the answers. – Glenn Stevens

Section and exercise numbers correspond to the online notes. Assignments are expected to be **neat** and **stapled**, with problems submitted **in the order they appear below**. **Illegible work may not be marked.**

Required problems for 3100 and 3100H

In the following problems, $\operatorname{lub} A$ denotes the least upper bound of the set A while $\operatorname{glb} A$ denotes its greatest lower bound. You are warned that outside of this class, it is more common to see $\sup A$ denoting the least upper bound (sup for “supremum”) and $\inf A$ denoting the greatest lower bound (inf for “infimum”).

1. Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences. Prove, directly from the definition of a Cauchy sequence, that $\{a_n + b_n\}$ is also Cauchy. **Do not take as known that Cauchy sequences converge**.

2. Let $\{a_n\}$ be a bounded increasing sequence. By the completeness axiom, we know $\{a_n\}$ converges to a real number limit.

   Show that in fact $\{a_n\}$ converges to $\operatorname{lub}\{a_n : n \in \mathbf{N}\}$.

   Don't be thrown off by the notation: $\{a_n\}$ denotes a sequence, while $\{a_n : n \in \mathbf{N}\}$ denotes the *set* of real numbers appearing as terms of that sequence.

3. Let S be a nonempty subset of $\mathbf{R}$ that is bounded below.

   (a) Let $S' = \{-s : s \in S\}$. Prove that S' is bounded above.

   (b) Let $U = \operatorname{lub} S'$. Show that $-U$ is the greatest lower bound of S.

   Hence, the LUB property of $\mathbf{R}$ implies the GLB property of $\mathbf{R}$.

4. Show that if A and B are nonempty sets of real numbers that are bounded above, and $A \subseteq B$, then $\operatorname{lub} A \leq \operatorname{lub} B$.

   Hint. There's a very short solution once you understand all the definitions.

5. In this exercise you will show that the sequence $\{\sin(n)\}$ does not converge.[1] An important role will be played by the identity

   $$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y), \tag{*}$$

   which you are familiar with from precalculus.

   Suppose for a contradiction that $\lim \sin(n) = L$ for the real number L.

   (a) Show that $\lim \cos(n) = L(\frac{1-\cos(1)}{\sin(1)})$.
   Hint. Start by taking $x = n$ and $y = 1$ in (*).

   (b) Show that $\lim \cos(n) = L(\frac{1-\cos(2)}{\sin(2)})$.

   (c) Comparing (a) and (b), deduce that $\lim \sin(n) = 0$ and $\lim \cos(n) = 0$.

   (d) Finish the proof by deriving a contradiction. *Hint.* What is $\sin^2 + \cos^2$?

[1] But by the Bolzano–Weierstrass theorem, it has a convergent subsequence!

MATH 3100H exercises

The following exercises introduce you to a concept in analysis known as the "limit superior" (or "limit supremum"). If you take MATH 4100, you will meet this notion again.

6. Let $\{a_n\}$ be a bounded sequence. For each natural number k, define the set

$$T_k = \{a_n : n \geq k\}.$$

We refer to T_k as the **k-tail set**: it is the collection of all real numbers that appear in the sequence at some index at least k.

Since $\{a_n\}$ is bounded above, each T_k is also bounded above. Thus, the Least Upper Bound property implies that each T_k has a least upper bound. We let L_k denote the least upper bound of T_k; that is,

$$L_k = \text{lub}\,\{a_n : n \geq k\}.$$

(So far you are being told all of this; you are not asked to prove the above facts.)

(a) Show that the sequence $L_1, L_2, L_3, \ldots$ is decreasing.

(b) Show that if V is a lower bound on $\{a_n\}$, then V is also a lower bound on $\{L_k\}$.

(c) Quickly explain why (a) and (b) imply that $\{L_k\}$ converges.

Remark. The limit of the sequence $\{L_k\}$ in part (c) is denoted "$\limsup a_n$". That is,

$$\limsup a_n = \lim \text{lub}\,\{a_n : n \geq k\}.$$

(This looks less weird when you remember that "sup" is commonly used in place of "lub.")

7. (continuation) Let $\{a_n\}$ be a bounded sequence and let $L = \limsup a_n$. That is, $L = \lim L_k$, where the numbers L_k are defined as in the last problem.

(a) Explain why L is a lower bound on $\{L_k\}$. You may cite results mentioned previously in class.

(b) Show that for every $\epsilon > 0$, and every natural number k, there is a natural number $n \geq k$ with $a_n > L - \epsilon$.

Hint. Could $L - \epsilon$ be an upper bound on $T_k = \{a_n : n \geq k\}$?

8. (continuation) Keep all notations and assumptions as in Exercises 6 and onwards.

(a) Let $\epsilon > 0$. Show that if k is a natural number with $L_k < L + \epsilon$, then $a_n < L + \epsilon$ for all $n \geq k$.

(b) Show that for every $\epsilon > 0$, there is an $K \in \mathbf{N}$ with $a_k < L + \epsilon$ for all natural numbers $k \geq K$.

9. (continution, and the BIG PAYOFF FOR ALL THESE EXERCISES) Keep all notations and assumptions as in Exercises 6 and onwards. Show that there is a subsequence of $\{a_n\}$ converging to $\limsup a_n$.

Remark. With a little more work, it can be proved that any convergent subsequence of $\{a_n\}$ converges to a number at most L. That is, $\limsup a_n$ is the largest limit of any convergent subsequence of $\{a_n\}$. Try showing this as practice!

Recommended problems (NOT to turn in)

§1.6: 9, 10, 12