MATH 3100 – Homework #6

posted October 24, 2025; due November 3, 2025

Section and exercise numbers correspond to the online notes (the ones in use earlier in the semester). Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.**

Required problems for all students

1. Suppose $\sum_{k=0}^{\infty} a_k x^k$ is a power series whose DOC has least upper bound R, where $0 < R < \infty$. Show that $\sum_{k=0}^{\infty} a_k x^k$ converges when $|x| < R$ and diverges when $|x| > R$.

   This completes the proof of the theorem from class characterizing the possible forms of the DOC. Needless to say, you should not assume that theorem in your proof!

2. §2.4: 1(b,e,f,k)

3. (a) Let f be a real-valued function with domain D and let $x_0 \in D$. The statement that f is continuous at x_0 can be written, in logical notation, as

   $$(\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x \in D)\ (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon).$$

   Write out the **negation** of this statement, again in logical notation.

   (b) Let f be the function defined by $f(x) = 0$ when $x \leq 0$ and $f(x) = 1$ when $x > 0$. Using (a), show that f is not continuous at $x_0 = 0$. Do *not* use the sequential criterion for continuity.

4. Let f be a real-valued function with domain D and let $x_0 \in D$. In class, we stated that continuity at x_0 and sequential continuity at x_0 are equivalent.

   We proved one direction: Continuity at x_0 implies sequential continuity at x_0. In this exercise you are asked to tackle the other direction. That is, supposing that f is not continuous at x_0, prove that f is not sequentially continuous at x_0.

   (a) Suppose f is not continuous at x_0. Show that there is some $\epsilon > 0$ with the following property: For every $n \in \mathbf{N}$, there is an $x_n \in D$ with $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \epsilon$.

   (b) Show that if the x_n are as in part (a), then $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$. (Hence, f is not sequentially continuous at x_0.)

5. (a) Show that for every pair of real numbers x and y,

   $$-|x - y| \leq |x| - |y| \leq |x - y|.$$

   Hint. Use the triangle inequality, writing $y = x + h$ for some h.

   (b) Show that $|x|$ is continuous. (We took this for granted earlier in the semester!)

6. Assume that the function "$\sqrt{x}$" makes sense. That is, there is a real-valued function labeled $\sqrt{x}$ defined for all $x \geq 0$ satisfying

(a) $\sqrt{x} \geq 0 \quad \text{for all } x \geq 0,$

(b) $(\sqrt{x})^2 = x \quad \text{for all } x \geq 0.$

(This follows from the Intermediate Value Theorem, to be treated shortly!)

Prove that $\sqrt{x}$ is continuous.

Suggestion. You may wish to prove continuity at $x_0 = 0$ separately from continuity at $x_0 > 0$.

7. Let $f(x) = 2x\cos(1/x)$ for $x \neq 0$ and set $f(0) = 0$. Show that f is continuous at 0 by directly verifying the ϵ-δ definition of continuity. Do *not* use the sequential criterion.

8. Prove that $f(x) = x^3$ is continuous at $x = 2$ by directly verifying the ϵ-δ definition of continuity. Do *not* use the sequential criterion.

Recommended problems

§2.3: 7, 10, 11
Ross: 17.5, 17.6

MATH 3100H problems

The following problems concern the notion of uniform continuity on a domain, which is stronger than ordinary continuity.

Suppose $f\colon D \to \mathbf{R}$. Recall that f is said to be continuous on D if f is continuous at each point $x_0 \in D$. Putting this in logical notation, saying f is continuous on D amounts to the requirement that

$$(\forall x_0 \in D)\ (\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x \in D)\ (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon).$$

What is uniform continuity and how does it compare? Loosely speaking, we say f is uniformly continuous on D if the δ in the previous definition can be chosen to depend **only** on ϵ, independently of $x_0 \in D$. More precisely, we say f is uniformly continuous on D if the following holds:

$$(\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x_0, x \in D)\ (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon).$$

Notice that the quantifier on x_0 has been moved further into the expression !

If you stare at this long enough, it should become clear that uniform continuity on D always implies continuity on D. (A δ that 'works' in the definition of uniform continuity also 'works' in the definition of continuity, and does so for all $x_0 \in D$ simultaneously.)

9. Let $f(x) = 1/x$ on the domain $D = (0, 1]$.

   (a) Show that for each $n \in \mathbf{N}$, we have $|\frac{1}{n} - \frac{1}{n+1}| < 1/n$ and $|f(\frac{1}{n}) - f(\frac{1}{n+1})| = 1$.

   Don't overthink this part!

   (b) Prove that f is **not** uniformly continuous on D.

   Suggestion. Consider $\epsilon = 1$ and use the result of (a).

10. In the last problem, we saw an example of a continuous function on $(0, 1]$ that is not uniformly continuous. In this problem, you will show that this cannot happen if $(0, 1]$ is replaced by the closed interval $[0, 1]$: Whenever $f\colon [0, 1] \to \mathbf{R}$ is continuous on $[0, 1]$, it must be that f is uniformly continuous on $[0, 1]$.

    Suppose for a contradiction that $f\colon [0, 1] \to \mathbf{R}$ is continuous but not uniformly continuous.

    (a) Show that there is some $\epsilon > 0$ for which the following holds: For every $n \in \mathbf{N}$, there are $x_n, y_n \in [0, 1]$ with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$.

    Hint. What is the negation of the definition of uniform continuity?

    (b) Choose sequences $\{x_n\}$, $\{y_n\}$ as in part (a). Explain why $\{x_n\}$ necessarily has a convergent subsequence whose limit belongs to $[0, 1]$.

    (c) Write the convergent subsequence from (b) in the form $\{x_{g(n)}\}$, where $g\colon \mathbf{N} \to \mathbf{N}$ is strictly increasing. Show that $\{y_{g(n)}\}$ is also convergent, and that in fact $\lim y_{g(n)} = \lim x_{g(n)}$.

    (d) Using (c) and the sequential criterion for continuity, show that $\lim f(x_{g(n)}) = \lim f(y_{g(n)})$.

    (e) Derive a contradiction to the result of (d) from our intiial choices of $\{x_n\}$ and $\{y_n\}$.

    Hint. What can you say about $|f(x_{g(n)}) - f(y_{g(n)})|$?