MATH 3100 – Homework #7

posted November 7, 2025; due November 14, 2025

Limits, like fears, are often just an illusion. – Michael Jordan

Required problems

1. Let $f\colon D \to \mathbf{R}$ be a constant function, meaning that there is a $c \in \mathbf{R}$ with $f(x) = c$ for all $x \in D$. Suppose $a \in \mathbf{R}$, and that for some $\delta > 0$, we have $(a - \delta, a + \delta) \setminus \{a\} \subseteq D$. Prove that $\lim_{x\to a} f(x) = c$.

   Hint. This problem is here to confirm you understand the definitions!

2. (Squeeze Lemma for sequences) Suppose $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are three sequences satisfying
$$a_n \le b_n \le c_n \quad \text{for all } n \in \mathbf{N}.$$
   Suppose there is a real number L such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Prove that $\lim_{n\to\infty} b_n = L$.

   Remark. This problem could have been assigned in Chapter 1; no new theory is required. The goal is to set you up to solve the next exercise.

3. (Squeeze Lemma for functions) Let f_1, f_2, and f_3 be three functions defined on all of $\mathbf{R}$, and let a be a real number. Suppose that
$$f_1(x) \le f_2(x) \le f_3(x) \quad \text{for all } x \in \mathbf{R}.$$
   Suppose also that there is a real number L for which
$$\lim_{x\to a} f_1(x) = \lim_{x\to a} f_3(x) = L.$$
   Prove that
$$\lim_{x\to a} f_2(x) = L.$$

4. Let $f : D \to \mathbf{R}$, and let $a \in \mathbf{R}$. Suppose that δ_1, δ_2 are positive real numbers for which $S_1 = (a - \delta_1, a + \delta_1) \setminus \{a\}$ and $S_2 = (a - \delta_2, a + \delta_2) \setminus \{a\}$ are both contained in D.

   Let $L \in \mathbf{R}$. Suppose that whenever $\{x_n\}$ is a sequence in S_1 for which $x_n \to a$, we have $f(x_n) \to L$. Prove that whenever $\{x_n\}$ is a sequence in S_2 for which $x_n \to a$, we have $f(x_n) \to L$.

   What's the point of this problem? A priori, one might worry that our definition of a functional limit allows $\lim_{x\to a} f(x)$ to take on two different values L and L', depending on the which δ the definition is invokved for. Maybe there's one value of $\delta > 0$ where the limit definition yields L as a limit, while there's a different $\delta' > 0$ for which the limit definition yields L' as a limit (with $L \ne L'$). Your work in this exercise shows that $\lim_{x\to a} f(x)$ cannot take two different real number limits. With a little more work, of the same kind, you could show that no two different limits from $\mathbf{R} \cup \{\pm\infty\}$ are possible.

5. Let $f : D \to \mathbf{R}$, and $g\colon E \to \mathbf{R}$. Suppose that a is a real number for which
$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M,$$
   where $L, M \in \mathbf{R}$. Prove that if $f(x) \le g(x)$ on the intersection $D \cap E$, then $L \le M$.

6. Let $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, and set $f(0) = 0$. Find, with proof, the value of $f'(0)$. Make sure to justify your evaluations of any limits using either the limit definition or known limit laws.

Recommended problems (from Ross)

§17: 11, 15
§20: 1, 9, 11, 13

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7. (Left and right-hand limits) Suppose $f\colon D \to \mathbf{R}$. Let $a \in \mathbf{R}$, and let $L \in \mathbf{R} \cup \{\pm\infty\}$. We say
$$\lim_{x \to a^+} f(x) = L$$
if the following conditions hold:

   (i) there is a $\delta > 0$ with $S^+ = (a, a+\delta) \subseteq D$,

   (ii) whenever $\{x_n\}$ is a sequence in S^+ for which $x_n \to a$, we have $f(x_n) \to L$.

   Similarly, we say
$$\lim_{x \to a^-} f(x) = L$$
if the following conditions hold:

   (i′) there is a $\delta > 0$ with $S^- = (a - \delta, a) \subseteq D$,

   (ii′) whenever $\{x_n\}$ is a sequence in S^- for which $x_n \to a$, we have $f(x_n) \to L$.

   (a) Prove that if $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

   (b) Now show the converse, in the case when L is a real number: If $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

   The assumption that L is a real number is not necessary for the result to hold, but it makes the proof a little less painful.

8. Suppose $f\colon D \to \mathbf{R}$. Let $a \in \mathbf{R}$, and suppose there is a real number $\delta > 0$ for which $(a - \delta, a + \delta) \subseteq D$. Prove that if $\lim_{x\to a} f(x) = f(a)$, then f is continuous at $x = a$.

   I suggest proving the contrapositive. Assume f is not continuous at $x = a$ and disprove that $\lim_{x\to a} f(x) = f(a)$. For the latter, it suffices to show (why?) that there is a sequence $\{x_n\}$ in $(a - \delta, a + \delta) \setminus \{a\}$ for which $f(x_n) \not\to f(a)$. Produce such a sequence by negating the ϵ-δ definition of continuity.