

MATH 3100 – Learning objectives to meet for Exam #2

The exam will cover §1.6–§2.3 of the course notes, as covered in lecture by end-of-class on Monday, 3/18. §1.7 was not discussed in class and is not examinable.

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- Completeness property of the real numbers
- Cauchy sequence
- Least upper bound of a set of real numbers
- Sequence of partial sums of an infinite series
- Convergence of an infinite series
- Geometric series
- Absolute and conditional convergence

Big theorems

Be familiar with all of the following theorems, making sure that you can give full statements for those not already spelled out below. For results proved in class, describe the components and main ideas of the proof.

- Every sequence has a monotone subsequence.
- Every bounded sequence has a convergent subsequence.
- Cauchy sequences are bounded.
- A sequence converges if and only if it is a Cauchy sequence.
- Every nonempty subset of the real numbers that is bounded above has a least upper bound.
- k th term divergence test (i.e., Proposition 2.1.10)
- Sum rule and constant multiple rule for series (Proposition 2.1.12, proved in HW)
- Convergence properties of geometric series (Proposition 2.1.14)
- Fundamental Principle of Nonnegative Series: A series with nonnegative terms converges if and only if its sequence of partial sums is bounded above.
- Comparison test
- Eventual comparison test

- Limit comparison test
- Integral comparison test
- Absolute convergence implies convergence
- Alternating series test
- Ratio test

What to expect on the exam

You can expect 5 questions on this exam. These include...

- At least one problem testing you on a basic definition and requiring you to give a proof using this definition (such as Practice Problem #1 below).
- At least one problem requiring you to apply the reasoning behind the integral test. See Problem #3 below.
- At least one multipart problem asking you to test concrete examples of series for convergence/divergence, and possibly to decide absolute or conditional convergence. See Problem #5 below.

Practice problems

- If $\sum_{k=1}^{\infty} a_k$ is an infinite series, what do we mean by the associated **sequence** $\{s_n\}$ **of partial sums**?
 - Say that the series $\sum_{k=1}^{\infty} a_k$ **diverges to infinity** if the sequence of partial sums $\{s_n\}$ diverges to infinity. Give a careful proof that if $\sum_{k=1}^{\infty} a_k$ diverges to infinity, then $\sum_{k=1}^{\infty} \frac{1}{2}a_k$ also diverges to infinity.
- Which sets of real numbers have a least upper bound? (No justification required.)
 - Let A be a set of real numbers with $\text{lub } A = 1$. Define

$$S = \{a + a' : a, a' \in A\}.$$

That is, S consists of all sums of two (possibly equal) elements of A . Give a careful proof that $\text{lub } S = 2$.

- Let f be a function that is nonnegative and *increasing* for $x \geq 1$.

- Explain why

$$f(n) \leq \int_n^{n+1} f(t) dt$$

for every natural number n . Your argument should mirror a step in our proof of the integral test.

- Using the result of (a), derive that $\sum_{n=1}^{99} n^2 \leq 333333$.

4. Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series with all nonnegative terms. Define

$$c_k = \begin{cases} a_k & \text{if } k \text{ is odd,} \\ b_k & \text{if } k \text{ is even.} \end{cases}$$

Prove that $\sum_{k=1}^{\infty} c_k$ converges.

5. For each of the following series, indicate whether it converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$,

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{e^{1/n}}{n^2}$,

(c) $\sum_{k=1}^{\infty} n(-3/5)^n$,

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1+\cos(n)}$.