MATH 3100 – Learning objectives to meet for Exam #3

The exam will cover §2.4 of the course notes, the portions of sections 17 (Continuous Functions), 20 (Limits of Functions), 28 (Basic Properties of the Derivative), and 29 (The Mean Value Theorem) from the Ross text that were treated in lecture, and our subsequent discussion of manipulating power series. No material from after Monday, April 22 is examinable.

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- power series
- domain of convergence and radius of convergence
- continuity of a function f at x_0
- f is continuous (with no point specified)
- ckosure of a subset $S \subseteq \mathbf{R}$
- $\lim_{x \to a^S} f(x)$
- $\lim_{x \to a} f(x)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, where a is a real number
- $\lim_{x \to -\infty} f(x)$, $\lim_{x \to \infty} f(x)$
- f is differentiable at a

Big theorems

Be familiar with all of the following theorems, making sure that you can give full statements for those not already spelled out below. For results proved in class, describe the components and main ideas of the proof.

- Ratio test
- Key Lemma: If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$, it converges absolutely for all x with $|x| < |x_0|$
- Characterization of the domain of convergence of a power series (Proposition 2.4.4)
- Sequential criterion for continuity
- Constant multiples, sums, products, and ratios of continuous functions are continuous
- The composition of continuous functions is continuous
- Intermediate Value Theorem
- Maximum Value Theorem

- Limit laws for sums, products, and quotients
- Differentiability at a implies continuity at a
- Derivatives of constant multiples, sums, products, and quotients
- Chain rule
- Rolle's theorem
- Mean Value Theorem
- If f has a maximum at $x_0 \in (a, b)$, and f is differentiable at x_0, then $f'(x_0) = 0$. And similarly for minimum.
- Differentiation and integration theorems for power series

What to expect on the exam

There will be five questions on the exam, possibly having multiple parts. These will include...

- A problem testing you on a basic definition and requiring you to give a proof using this definition
- A problem or problem part asking you to determine the domain and/or radius of convergence of (one or more) power series
- A problem requiring you to apply our differentiation and/or integration rules for manipulating power series

Practice problems

1. Find the domain of convergence of $\displaystyle\sum_{n=0}^{\infty}(-2/3)^n x^n$. Do the same for $\displaystyle\sum_{n=1}^{\infty}\frac{1}{5^n n}x^n$. Justify your answers.

2. (a) What does it mean to say that f is continuous at x_0? Make sure to state all of the assumptions on f.

   (b) Prove, directly from the ϵ-δ definition, that $f(x) = 1/x^2$ is continuous at $x = 1$. You may **not** use the sequential criterion for continuity or the continuity rules proved in class.

3. Suppose that f is a real-valued function defined on $(-\infty, \infty)$ with the property that $f(x) > 0$ for every real number x. Prove that if $\lim_{x\to\infty} f(x)$ exists as a real number L, then $L \geq 0$. Must we have $L > 0$?

4. (a) What does it mean to say that $f(x)$ is differentiable at $x = c$?

   (b) State the Mean Value Theorem.

   (c) Suppose f is differentiable on all of $\mathbf{R}$ and that there is no $x \in \mathbf{R}$ with $f'(x) = 1$. Prove that there is at most one number x with $f(x) = x$.

5. (a) Find a power series representing $\frac{1}{1+x^2}$ for $x \in (-1, 1)$.

   (b) Find a power series representing $\int_0^x \frac{1}{1+t^2}\, dt$ for $x \in (-1, 1)$.

   (c) Find the exact value of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

   Simplify your answer as much as possible.

   Hint: Recall from calculus that $\int_0^x \frac{1}{1+t^2}\, dt = \arctan(x)$.

6. Let f be a continuous function on $[0, 1]$. Suppose that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$, with $f(0) > 0$ and $f(1) < 1$. Use the Intermediate Value Theorem to prove that the graph of $y = f(x)$ on the interval $[0, 1]$ intersects the graph of $y = x$ on $[0, 1]$.