MATH 3100 – Homework #1

posted January 12, 2024; due January 22 by end-of-day

It requires a very unusual mind to undertake the analysis of the obvious. – A.N. Whitehead

Section and exercise numbers correspond to the notes of Dr. Adams. Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.**

1. §1.1: Exercise 4.
2. §1.1: Exercise 5.
3. §1.1: Exercise 8.
4. §1.2: Exercise 5.
5. §1.2: Exercise 10.
6. §1.2: Exercise 14(b).
7. The following is a statement of complete induction with a different base case:

    Suppose $S \subseteq \mathbf{N}$. Let $n_0 \in \mathbf{N}$, and suppose both of the following hold:

    (i) $n_0 \in S$,

    (ii) if n is a natural number with $n \geq n_0$, and all of $n_0, n_0+1, \ldots, n \in S$, then $n+1 \in S$.

    Then $S \supseteq \{n \in \mathbf{N} : n \geq n_0\}$.

    We will take this as a basic principle of mathematical reasoning.

    In class we considered the following statement: *For every $n \in \mathbf{N}$ with $n \geq 12$, one can make n cents postage out of 4 cent and 5 cent stamps.* Fill in the details of the following proof. For your submission, you are expected to **write out the complete argument on your own sheet of paper**!

    Let $S = \{n \in \mathbf{N} : \text{one can make } n \text{ cents postage out of 4 and 5 cent stamps}\}$. We want to show that $S \supseteq \{n \in \mathbf{N} : n \geq 12\}$. We apply complete induction with base case $n_0 = 12$.

    First, $12 \in S$, since [fill this in!].

    Now let $n \in \mathbf{N}$ where $n \geq 12$, and assume that all of $12, 13, \ldots n \in S$. We will show $n+1 \in S$. If $n = 12$, 13, or 14, then $n+1 \in S$ since [fill this in !].

    Thus, we can assume $n \geq 15$. Then $n+1 \geq 16$, and $(n+1) - 4 \geq 12$. Therefore, [fill this in!].

    Hence, $n+1 \in S$. By complete induction, S contains all natural numbers $n \geq 12$, as desired.

8. §1.2: Exercise 19.
9. Define real numbers α and β by $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

    (a) Check that α and β are roots of the polynomial $x^2 - x - 1$.

(b) Using (a), deduce that $\alpha^{n+1} = \alpha^n + \alpha^{n-1}$ and $\beta^{n+1} = \beta^n + \beta^{n-1}$, for every integer n. (First use (a) to explain why this holds when $n = 1$. Then deduce the general case. For the general case you don't need induction, just algebra!)

(c) Recall that the Fibonacci sequence $\{F_n\}$ is defined by $F_1 = 1$, $F_2 = 1$, and the recurrence $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.
Use complete induction to prove that $\frac{\alpha^n - \beta^n}{\sqrt{5}} = F_n$ for all natural numbers n.
Hint: The result of (b) will be useful.

10. The following argument is an *alleged* proof that in any finite group of people, all of them have the same height:

> Let S be the set of natural numbers n for which the statement "every group of n people share the same height" is true. Obviously the statement is true if there is just one person, so $1 \in S$. Now we suppose that $n \in S$, and we prove that $n + 1 \in S$. Take any group of $n + 1$ people, say $A_1, \ldots, A_{n+1}$. Since $n \in S$, it must be that $A_1, \ldots, A_n$ all share the same height, and similarly for $A_2, \ldots, A_{n+1}$. But these two groups overlap; for instance, the second person A_2 is in both. So all of our $n + 1$ people have the same height (in fact, everyone is the same height as A_2). Thus, $n + 1 \in S$. So by induction, S is all of the natural numbers.

Clearly explain the mistake in the proof.