**MATH 3100 – Homework #4**

posted February 16, 2024; due February 23, 2024

Answer the questions, then question the answers. – Glenn Stevens

Section and exercise numbers correspond to the online notes. Assignments are expected to be **neat** and **stapled**, with problems submitted **in the order they appear below**. **Illegible work may not be marked.**

## Required problems

1. §1.5: 6

   In the following problems, $\operatorname{lub} A$ denotes the least upper bound of the set $A$ while $\operatorname{glb} A$ denotes its greatest lower bound. You are warned that outside of this class, it is more common to see $\sup A$ denoting the least upper bound (sup for "supremum") and $\inf A$ denoting the greatest lower bound (inf for "infimum").

2. Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences. Prove, directly from the definition of a Cauchy sequence, that $\{a_n + b_n\}$ is also Cauchy. **Do not assume that Cauchy sequences converge**.

3. Let $\{a_n\}$ be a bounded increasing sequence. By the completeness axiom, we know $\{a_n\}$ converges to a real number limit.

   Show that in fact $\{a_n\}$ converges to $\operatorname{lub}\{a_n : n \in \mathbf{N}\}$.

   Don't be thrown off by the notation: $\{a_n\}$ denotes a sequence, while $\{a_n : n \in \mathbf{N}\}$ denotes the *set* of real numbers appearing as terms of that sequence.

4. Let $S$ be a nonempty subset of $\mathbf{R}$ that is bounded below.

   (a) Let $S' = \{-s : s \in S\}$. Prove that $S'$ is bounded above.

   (b) Let $U = \operatorname{lub} S'$. Show that $-U$ is the greatest lower bound of $S$.

   Hence, the LUB property of $\mathbf{R}$ implies the GLB property of $\mathbf{R}$.

5. Show that if $A$ and $B$ are nonempty sets of real numbers that are bounded above, and $A \subseteq B$, then $\operatorname{lub} A \leq \operatorname{lub} B$.

   *Hint.* There's a very short solution once you understand all the definitions.

6. Let $\{a_n\}$ be a sequence that is bounded above. For each natural number $k$, define the set

   $$T_k = \{a_n : n \geq k\}.$$

   We refer to $T_k$ as the **$k$-tail set**: it is the collection of all real numbers that appear in the sequence at some index at least $k$.

   Since $\{a_n\}$ is bounded above, each $T_k$ is also bounded above. Thus, the Least Upper Bound property implies that each $T_k$ has a least upper bound. We let $L_k$ denote the least upper bound of $T_k$; that is,

   $$L_k = \operatorname{lub}\{a_n : n \geq k\}.$$

   (So far you are being told all of this; you are not asked to prove the above facts.)

(a) Show that the sequence $L_1, L_2, L_3, \ldots$ is decreasing.

(b) Show that if $V$ is a lower bound on $\{a_n\}$, then $V$ is also a lower bound on $\{L_k\}$.

(c) Quickly explain why (a) and (b) imply that $\{L_k\}$ converges.

*Remark.* The limit of the sequence $\{L_k\}$ in part (c) is denoted "$\limsup a_n$". That is,

$$\limsup a_n = \lim \operatorname{lub} \{a_n : n \geq k\}.$$

(This looks less weird when you remember that sup is commonly used in place of lub.)

7. (continuation) Let $\{a_n\}$ be sequence that is bounded above and let $L = \limsup a_n$. That is, $L = \lim L_k$, where the numbers $L_k$ are defined as in the last problem.

   (a) Explain why $L$ is a lower bound on $\{L_k\}$. You may cite results mentioned previously in class.

   (b) Show that for every $\epsilon > 0$, and every natural number $k$, there is a natural number $n \geq k$ with $a_n > L - \epsilon$.

   Hint. Could $L - \epsilon$ be an upper bound on $T_k = \{a_n : n \geq k\}$?

8. (continuation) Keep all notations and assumptions as in Exercises 6 and onwards.

   (a) Let $\epsilon > 0$. Show that if $k$ is a natural number with $L_k < L + \epsilon$, then $a_n < L + \epsilon$ for all $n \geq k$.

   (b) Show that for every $\epsilon > 0$, there is an $K \in \mathbf{N}$ with $a_k < L + \epsilon$ for all natural numbers $k \geq K$.

9. (continution, and the BIG PAYOFF FOR ALL THESE EXERCISES) Keep all notations and assumptions as in Exercises 6 and onwards. Show that there is a subsequence of $\{a_n\}$ converging to $\limsup a_n$.

   *Remark.* With a little more work, it can be proved that any convergent subsequence of $\{a_n\}$ converges to a number at most $L$. <u>That is, $\limsup a_n$ is the largest limit of any convergent subsequence of $\{a_n\}$.</u> Try showing this as practice! The lim *sup* is a big deal if you go in in real analysis.

## Recommended problems (NOT to turn in)

§1.6: 9, 10, 12