MATH 3200 – Homework #2

posted August 27, 2024; due at the **start of class** on September 4, 2024

Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.** Assume — unless explicitly told otherwise — that **you are expected to include clear and concise explanations of your reasoning, expressed in complete sentences.**

For the next series of exercises, you are expected to use the following definition.

Definition. Let m and n be integers. We say that m **divides** n if $n = mk$ for some integer k. In that case, we write $m \mid n$, and we say that n is a **multiple** of m.

For example, -3 divides 6, since $6 = (-3)k$ for the integer $k = -2$. Make sure to get the order the right way around: -3 divides 6, but 6 *does not* divide -3. So it is correct to write $-3 \mid 6$, but incorrect to write $6 \mid -3$.

1. Prove or disprove: If a is an integer, then a divides a.

2. Prove or disprove: Every integer divides 0.

3. Prove or disprove: 0 divides every integer.

4. Prove or disprove: If a, b, c, and d are integers such that $a \mid b$ and $c \mid d$, then $ac \mid bd$.

5. Determine the negation of each of the following. In each case, try to move the "not" as far into the sentence as possible. Afterwards, give a concise one-line proof or disproof of the original statement.

   (a) Every integer is either even or divisible by 3.

   (b) There is an integer that is both even and divisible by 3.

   (c) There is no integer that divides every even integer.

   (d) $\forall x, y \in \mathbf{Z}$, $\exists z \in \mathbf{Z}$ such that $x \mid z$ and $y \mid z$.

   [Here $\mathbf{Z}$ is the collection of all integers, meaning $0, \pm 1, \pm 2, \pm 3, \ldots$.]

6. Prove or disprove: Let a, b, c be integers. If $a \nmid bc$, then $a \nmid b$.

7. Prove or disprove: Let a and b be integers. If ab and $a + b$ are even, then both a and b are even.

   [Assume that all integers are even or odd, and that no integer is both.]

8. Prove or disprove: An integer is even if and only if it can be expressed as a sum of two odd integers.

   ["P if and only if Q" means "P implies Q **and** Q implies P". So to prove the above statement, you would need to show that if x is an even integer, then x is a sum of two odd integers, **and** that if x is a sum of two odd integers, then x is even.]

For the next two exercises, we recall our BASIC RULES/AXIOMS concerning inequalities. For all real numbers a, b, c, we assume:

- R0. $1 \neq 0$
- R1. If $a > b$ and $b > c$, then $a > c$.
- R2. If $a > b$ then $a + c > b + c$.
- R3. If $a > b$ and $c > 0$, then $ac > bc$.
- R4. If $a > b$ and $c < 0$, then $ac < bc$.
- R5. For every real number a, **exactly one** of the following is true: $a > 0$, $a = 0$, or $a < 0$.

9. Let x be a real number. Show that if $x > 0$, then $\frac{1}{x} > 0$. Explicitly say where and how you use each of the rules above.

10. Let x be a real number. Show that if $x \neq 0$, then $x^2 > 0$. Explicitly say where and how you use each of the rules above.