MATH 3200 – Homework #3

posted September 18, 2024; due at the **start of class** on Wednesday, September 25

Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.** Assume — unless explicitly told otherwise — that **you are expected to include clear and concise explanations of your reasoning, expressed in complete sentences.**

For each of the statements below, supply a proof using some form of the induction priciple. In each case, make sure to identify the statements $P(n)$ being proven! See the "proof" in Exercise 9 below for a sample of how to work this in to your write-ups.

1. The sum of the first n even positive integers is $n^2 + n$.

2. The sum of the cubes of the first n positive integers is $(\frac{n(n+1)}{2})^2$.

3. $(1 + \frac{1}{2024})^n \geq 1 + \frac{n}{2024}$ for all nonnegative integers n.

   [We stated induction where the base case corresponded to $n = 1$. But the induction principle is valid with any integer you like serving as the base case.

   For this problem, start your numbering with 0 instead of 1: That is, your statements are now numbered $P(0), P(1), \ldots$. Your base case is now $P(0)$. And in the induction step, you should prove that if n is an integer at least 0, and $P(n)$ is true, then $P(n + 1)$ is true.]

4. Prove that $n! > 3^n$ for all natural numbers $n \geq 7$.

   [For this problem, start your numbering with 7 instead of 1: That is, your statements are now numbered $P(7), P(8), \ldots$. Your base case is now $P(7)$. And in the induction step, you should prove that if n is an integer at least 7, and $P(n)$ is true, then $P(n + 1)$ is true.]

5. Define real numbers α and β by $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

   (a) Check that α and β are roots of the polynomial $x^2 - x - 1$.

   (b) Using (a), deduce that $\alpha^{n+1} = \alpha^n + \alpha^{n-1}$ and $\beta^{n+1} = \beta^n + \beta^{n-1}$, for every integer n. (This part is *not* to be solved by induction. Use (a) to deduce the equality for $n = 1$. Then use algebra...)

   (c) Recall that the Fibonacci sequence $\{F_n\}$ is defined by $F_1 = 1$, $F_2 = 1$, and the recurrence $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.
   Use complete induction to prove that $\frac{\alpha^n - \beta^n}{\sqrt{5}} = F_n$ for all natural numbers n.

   [*Hint:* The result of (b) will be useful.]

6. Consider the following statement.

   For every natural number n, the number $n^2 + n + 41$ is a prime number.

   Either give a counterexample, or prove the statement by induction.

7. Let $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and define $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Then $a_n < 2^n$ for all positive integers n.

8. Every positive integer can be written as a sum of distinct powers of 2.

   [*Hint*: Here the powers of 2 are $2^0, 2^1, 2^2, \ldots$. In particular, $1 = 2^0$ counts as a power of 2.

   In the induction step, when you assume $P(1), \ldots P(n)$ and try to prove $P(n + 1)$, it may help to take two cases: $n + 1$ is even, or $n + 1$ is odd.]

9. The following argument is an *alleged* proof by induction that any finite group of people all have the same height:

> For every natural number n, let $P(n)$ be the statement "every group of n people share the same height".
>
> Base case: $P(1)$ is true, since if there is just one person, they all have the same height!
>
> Induction step: We now suppose that $P(n)$ is true and we prove $P(n+1)$. Consider any group of $n+1$ people, say $A_1, \ldots, A_{n+1}$. Since $P(n)$ holds, it must be that $A_1, \ldots, A_n$ all share the same height, and similarly for $A_2, \ldots, A_{n+1}$. But these two groups overlap; for instance, the second person A_2 is in both. So all of our $n+1$ people have the same height (indeed, everyone is the same height as A_2). Thus, $P(n+1)$ holds.
>
> By induction, $P(n)$ is true for all natural numbers n.

Exactly where is the mistake in this proof?