MATH 3200 – Homework #6

posted November 6, 2024; due at the **start of class** on November 13, 2024

Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.**

Assume — unless explicitly told otherwise — that **you are expected to include clear and concise explanations of your reasoning, expressed in complete sentences.**

1. (a) Let R be the divisibility relation on the set $\mathbf{Z}$ of integers. That is,

$$R = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} : a \mid b\}.$$

What is the domain of R? What is the range of R? (Keep in mind the instructions: Explain your answers!)

(b) Let R be the relation on the set $\mathbf{R}$ of real numbers defined by $R = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = x^2\}$. What is the domain of R? What is the range of R?

(c) Let R be the relation on the set $\mathbf{R}$ of real numbers defined by

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 = 9\}.$$

What is the domain of R? What is the range of R?

2. Reflexive Ralph asserts that our definition of an equivalence relation is redundant, because reflexivity follows from symmetry and transitivity. Here is his argument.

Claim: If R is a relation on a set S, and R is both symmetric and transitive, then R is reflexive.

Proof: Suppose R is a relation on S that is symmetric and transitive. Let x be any element of S. As R is symmetric, $x\,R\,y$ implies that $y\,R\,x$. Since $x\,R\,y$ and $y\,R\,x$, transitivity implies that $x\,R\,x$. Since this holds for all $x \in R$, we have shown that R is reflexive.

Ralph's claim is not correct: In class, we saw multiple examples of relations on $\{A, B, C, D, E\}$ that were symmetric and transitive but not reflexive. Identify the flaw in Ralph's "proof".

3. Let R be the relation on the set $\mathbf{Z}$ of integers defined by declaring $x\,R\,y$ when 3 divides $x + y$. That is,

$$R = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} : 3 \mid x + y\}.$$

Is R reflexive? Is R symmetric? Is R transitive?

4. Let R be the relation on the set $\mathbf{R}$ of real numbers defined by $x\,R\,y$ when $x - y \in \mathbf{Z}$. That is,

$$R = \{(x, y) \in \mathbf{R}^2 : x - y \in \mathbf{Z}\}.$$

Prove that R is an equivalence relation by verifying that R is reflexive, symmetric, and transitive.

5. Let $\sim$ be an equivalence relation on S. Let $\mathscr{C}$ denote the collection of all equivalence classes under $\sim$; that is,

$$\mathscr{C} = \{[x] : x \in S\}.$$

Prove that

$$\bigcup_{C \in \mathscr{C}} C = S.$$

Hint. The proof is short!

6. A relation R on a set S is called **circular** if the following holds: For all $x, y, z \in S$, if $x \, R \, y$ and $y \, R \, z$, then $z \, R \, x$. Prove that if R is an equivalence relation on S, then R is circular.

7. (continuation) Suppose that R is a relation on S that is both reflexive and circular. Prove that R is an equivalence relation on S. Make sure it is clear exactly where and when you invoke the reflexivity and circularity hypotheses.

8. (Set partitions determine equivalence relations) Let S be a set. A **partition of** S is a set $\mathscr{C}$ of subsets of S with the following three properties:

   (a) Every $C \in \mathscr{C}$ is nonempty.

   (b) $\bigcup_{C \in \mathscr{C}} C = S$,

   (c) For all $C_1, C_2 \in \mathscr{C}$, either $C_1 = C_2$ or $C_1 \cap C_2 = \emptyset$.

   For example, if S is the five-element set $S = \{A, B, C, D, E\}$, one partition of S is $\mathscr{C} = \{\{A, B\}, \{C, D\}, \{E\}\}$.

   Suppose that $\mathscr{C}$ is a partition of the set S. Define a relation $\sim$ on S by declaring $x \sim y$ when there exists a set $C \in \mathscr{C}$ with $x \in C$ and $y \in C$. Prove that $\sim$ is an equivalence relation on S.

9. (Tricky!) Let $S = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} : y \neq 0\}$. Let $\sim$ be the relation on S defined by $(a, b) \sim (c, d)$ precisely when $ad = bc$. That is,

$$R = \{((a, b), (c, d)) \in S \times S : ad = bc\}.$$

   (This is a bit confusing at first, because not only are elements of R ordered pairs — as we are used to — but elements of S are also ordered pairs!) Prove that $\sim$ is an equivalence relation on S.