$$N = \square + \square + \square$$

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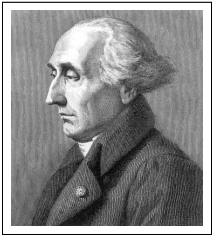
Characterizing sums of squares

The study of sums of squares goes back at least to the dawn of modern number theory.

Let $\square$ stand for a generic member of the set $\{n^2 : n = 0, 1, 2, \dots\}$.

Theorem (Fermat–Euler)

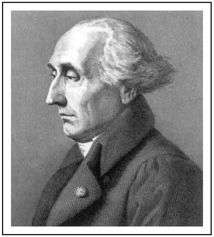
Let n be a natural number. Then $n = \square + \square$ if and only if every prime p dividing n with $p \equiv 3 \pmod 4$ shows up to an even power.



Theorem (Lagrange)

Every natural number is of the form $\square + \square + \square + \square$.

We teach both results in courses on elementary number theory. But what about 3 squares?



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Theorem (Legendre)

Let n *be a natural number. Then* n *has the form* $\square + \square + \square$ *unless* $n = 4^k(8l + 7)$ *for some nonnegative integers* k *and* l.

How is the three-squares theorem proved?

First, change the problem. Ask about three *rational squares*.
By Hasse-Minkowski,

$$N = \square + \square + \square \text{ in } \mathbb{Q} \iff N = \square + \square + \square \text{ in each } \mathbb{Q}_p.$$

Lemma
If $N = \square + \square + \square$ in $\mathbb{Q}$, then $N = \square + \square + \square$ in $\mathbb{Z}$.

For details, see
Serre's *Course in Arithmetic*, appendix to Chapter 4.

Counting sums of squares

Theorem (I. M. Trivial)

$$\#\{n \leq x : n = \square\} = \sqrt{x} + O(1).$$

Theorem (Landau–Ramanujan)

As $x \to \infty$,

$$\#\{n \leq x : n = \square + \square\} \sim C \frac{x}{\sqrt{\log x}},$$

where

$$C = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{-1/2}.$$

Theorem

For $x \geq 2$, we have

$$\#\{n \leq x : n = \Box + \Box + \Box\} = \frac{5}{6}x + O(\log x).$$

Proof.

Let's count exceptions.

$$\#\{n \leq x : n \equiv 7 \pmod 8\} = \frac{x}{8} + O(1).$$

$$\#\{n \leq x : n = 4m, m \equiv 7 \pmod 8\} = \frac{x}{8 \cdot 4} + O(1),$$

etc. Notice that $1/8 + 1/(8 \cdot 4) + 1/(8 \cdot 4^2) + \cdots = 1/6$.

Was that too easy?

Let $\Delta(x)$ denote the remainder term in the counting problem for three squares.

Theorem (Osbaldestin and Shiu)

There is a continuous, nowhere differentiable function F with period 1 such that for $N \geq 1$,

$$\frac{1}{N}\sum_{0\leq n<N}\Delta(n) = \frac{3}{8}\frac{\log N}{\log 4} + F\left(\frac{\log N}{\log 4}\right) + \frac{\delta(N)}{N},$$

where

$$\delta(N) = \begin{cases} \frac{1}{8} & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even.} \end{cases}$$

A sampling of probabilistic number theory

Let f be an arithmetic function. The mean value of f is

$$\mathcal{M}(f) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n),$$

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- If $f(n) = \mathbf{1}_S$, then $\mathcal{M}(f)$ is the asymptotic density of S.
- If $f(n) = \mu(n)$, then f has mean value 0. This is equivalent to the PNT.

Which f have mean values?

For this to be a reasonable question, we should impose some conditions on f. We will assume:

1. f is multiplicative,
2. f is real-valued,
3. f takes values in $[-1, 1]$.

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Starting point: Write $f(n) = \sum_{d|n} g(d)$ for some arithmetic function g. Then

$$\sum_{n \le x} f(n) = \sum_{n \le x} \sum_{d|n} g(d) = \sum_{d \le x} g(d) \sum_{\substack{n \le x \\ d|n}} 1.$$

A good guess?

This suggests that

$$\frac{1}{x}\sum_{n\le x} f(n) \approx \sum_{d\le x}\frac{g(d)}{d}.$$

So taking the limit, we might expect

$$\begin{aligned}\mathcal{M}(f) &= \sum_{d}\frac{g(d)}{d}\\ &= \prod_{p}\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\dots\right).\end{aligned}$$

Also, g is multiplicative and $g(p^k) = f(p^k) - f(p^{k-1})$.

Conjecture

Under our hypotheses on f, we have

$$\mathcal{M}(f) = \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \sum_{m=1}^{\infty} f(p^m) p^{-m}\right)$$

Theorem (Wintner)

The conjecture holds if

$$\sum_p \frac{1 - f(p)}{p}$$

converges.

Example

The function $f(n) = \phi(n)/n$ has a mean value.

Conjecture (Erdős, Wintner)

Assume f is as before. If

$$\sum_p \frac{1-f(p)}{p}$$

diverges, then $\mathcal{M}(f) = 0$.

Because of the case $f = \mu$, this include the prime number theorem.

From Elliott's *Probabilistic Number Theory, vol. 1*:

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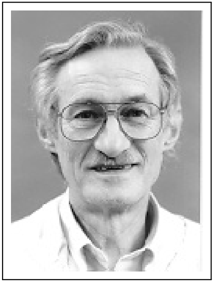
From the errata:

> *The prize should read* $10^{10!}$.

Truth and consequences



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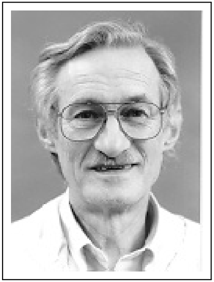

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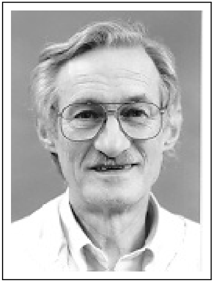

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But what is this theorem good for, besides earning prize money?

Return to sums of three squares

Let ϕ denote Euler's totient function, so that

$$\phi(n) = \#\{1 \le k \le n : \gcd(k, n) = 1\}.$$

Question: How often is $\phi(n)$ a sum of squares?

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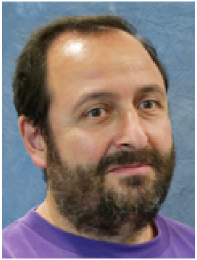
Theorem (Banks, Luca, Saidak Shparlinski)

For large x,

$$\#\{n \leq x : \phi(n) = \square + \square\} \asymp \frac{x}{(\log x)^{3/2}}.$$

Bill Banks, Igor Shparlinski, Filip Saidak, and John Friedlander

Three squares?

Theorem (P.)

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The set of n for which $\phi(n)$ is a sum of three squares has density $7/8$.

Let $v_2(m)$ be the power of 2 sitting inside m, and let $u(m)$ be the odd part of m, so that

$$m = 2^{v_2(m)} u(m).$$

According to Legendre,

$$\begin{aligned}\phi(n) \neq \square + \square + \square &\iff \phi(n) = 4^k(8l+7) \text{ for some } k, l \\ &\iff 2 \mid v_2(\phi(n)), \quad u(\phi(n)) \equiv 7 \pmod 8\end{aligned}$$

Let G be the group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^{\times}$. Define a map $r\colon \mathbb{N} \to G$ by

$$m \mapsto (v_2(m) \bmod 2, u(m) \bmod 8).$$

Then r is a homomorphism (of semigroups).
Now define a map $f\colon \mathbb{N} \to G$ by

$$n \mapsto r(\phi(n)).$$

Then f is a G-valued multiplicative function, in the sense that whenever a and b are relatively prime,

$$f(ab) = f(a) \circ f(b).$$

Also, Legendre says

$$\phi(n) \neq \Box + \Box + \Box \iff f(n) = (0 \bmod 2, 7 \bmod 8).$$

We will show that as n ranges over $\mathbb{N}$, the elements $f(n) \in G$ become equidistributed. In other words, for each fixed $g \in G$, the set $f^{-1}(g) \subset \mathbb{N}$ has asymptotic density $1/8$.

Lemma

Let $g_1, g_2, g_3, \ldots$ be an infinite sequence of elements of a finite abelian group G. Then $\{g_i\}_{i=1}^{\infty}$ is uniformly distributed precisely when

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} \chi(g_n) = 0$$

for each nontrivial $\chi \in \hat{G}$.

So let χ be a nontrivial character of $G = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^{\times}$.
Every element of G squares to be the identity, so the image of χ belongs to $\{1, -1\}$.
We have to show that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} \chi(f(n)) = 0.$$

In other words, we want that the function $\chi \circ f$ has mean value zero.
We can apply Wirsing, because:

- $\chi \circ f$ is multiplicative,
- $\chi \circ f$ is real-valued,
- $\chi \circ f$ assumes only the values -1 and 1.

Wirsing says we should look at

$$\sum_{p} \frac{1-\chi(f(p))}{p} = 2 \sum_{p\colon\, \chi(f(p))=-1} \frac{1}{p}.$$

To show this diverges, we want many primes p for which $\chi(f(p)) = -1$.
The characters of $G = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^{\times}$ have the form $\chi_1\chi_2$:

Characters χ_1 of $\mathbb{Z}/2\mathbb{Z}$: $\{\mathbf{1}, (-1)^n\}$
Characters χ_2 of $(\mathbb{Z}/8\mathbb{Z})^{\times}$: Dirichlet characters mod 8

So for our χ, we have

$$(\chi \circ f)(n) = \zeta^{v_2(\phi(n))}\chi_2(u(n)),$$

where $\zeta \in \{1, -1\}$ and χ_2 is a Dirichlet character mod 8.
Also either $\zeta \neq 1$ or $\chi_2 \neq \mathbf{1}$.

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Suppose first $\chi_2 = \mathbf{1}$. Then $\zeta = -1$, and

$$\chi(f(p)) = (-1)^{v_2(p-1)}.$$

So if $p \equiv 3 \pmod 4$, then $\chi(f(p)) = -1$. So

$$\sum_{p\colon\ \chi(f(p))=-1} \frac{1}{p} = \infty.$$

This proves $\chi \circ f$ has mean value zero.

Now suppose $\chi_2 \neq \mathbf{1}$. Take a prime $p \equiv 5 \pmod 8$. We have $v_2(p-1) = 2$, and so

$$\begin{aligned} \chi(f(p)) &= \zeta^{v_2(p-1)} \chi_2(u(p-1)) \\ &= \chi_2(u(p-1)). \end{aligned}$$

Either $\chi_2(3) = -1$ or $\chi_2(5) = -1$. In the former case, choose p so that

$$\frac{p-1}{4} \equiv 3 \pmod 8, \qquad \text{i.e.,} \qquad p \equiv 13 \pmod{32}$$

and in the latter choose p so that

$$\frac{p-1}{4} \equiv 5 \pmod 8, \qquad \text{i.e.,} \qquad p \equiv 21 \pmod{32}.$$

Then $\chi(f(p)) = -1$. And the sum of the reciprocals of these p diverges. So $\chi \circ f$ has mean value zero in this case also.

What about other arithmetic functions?

A theorem of Ruzsa says that if G is any finite abelian group, and $f : \mathbb{N} \to G$ is a multiplicative function, then the sets $f^{-1}(g)$, with $g \in G$, all possess asymptotic densities.

Corollary

Let h be any positive-integer valued multiplicative function. Then all three of the sets

$$\{n : h(n) = \Box\}, \quad \{n : h(n) = \Box + \Box\}, \quad \{n : h(n) = \Box + \Box + \Box\}$$

have asymptotic densities.

A parting shot

Let $\lambda(n)$ denote the exponent of the group $(\mathbb{Z}/n\mathbb{Z})^\times$.

Theorem (P.)

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Conjecture

The set of n for which $\lambda(n)$ is a sum of three squares does not have an asymptotic density.