MATH 4000/6000 – Homework #1

posted January 12, 2024; due by end of day on January 22, 2024

> You know, for a mathematician, he did not have enough imagination. But he has become a poet and now he is fine. — David Hilbert (1862–1943), talking of an ex-student

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are *required* for those in MATH 6000 and *extra credit* for those in MATH 4000.

Fully explain your answers. In problems #1 and #2 you must explain which algebraic properties (properties A1–A4, M1–M3, D1 on the handout) you are using at every step of the proof. I recommend a "two column" format with each line showing, on the left, a step in the proof and on the right a justification. For example:

$$a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \qquad \text{(Distributive law).}$$

Refer to properties by <u>name</u> rather than <u>number</u> (write "Distributive law" and not "D1").

For all other problems, you are expected to do "familiar" algebraic manipulations without citing the algebraic properties on the handout. **Throughout this problem set, you may also assume that $a \cdot 0 = 0 = 0 \cdot a$ and $(-1)a = -a = a(-1)$ for all a, as already shown in class**. You may also assume that $a > 0$ is equivalent to $a \in \mathbb{Z}^+$.

1. Axiom A4 states that for every $a \in \mathbb{Z}$, there is a $b \in \mathbb{Z}$ with $a + b = 0$. Using the properties listed on the handout, prove that for every $a \in \mathbb{Z}$ there is a *unique* $b \in \mathbb{Z}$ with $a + b = 0$. (This uniqueness of b justifies the notation $-a$ for *the* additive inverse of a.)

2. Prove that for all $a, b \in \mathbb{Z}$, we have
   (a) $(-a)b = -(ab)$.
   (b) $(-a)(-b) = ab$.

3. (a) Let $a, b \in \mathbb{Z}$ and suppose that $a < b$. Prove that $a + c < b + c$ for every $c \in \mathbb{Z}$.
   (b) Let $a, b \in \mathbb{Z}$ and suppose that $a < b$. Prove that $ac < bc$ for every $c \in \mathbb{Z}^+$.

4. Let $a, b \in \mathbb{Z}$. Show that $a < b$, $a = b$, or $a > b$, and in fact exactly one of these three holds.

5. Let $a, b \in \mathbb{Z}$.
   (a) Prove that if $a < 0$ and $b < 0$, then $ab > 0$.
   (b) Show that if $a < 0$ and $b > 0$, then $ab < 0$.
   (c) Show that if $ab = 0$, then either $a = 0$ or $b = 0$.

6. Use the Well-Ordering Principle to prove the following version of the Principle of Mathematical Induction.

   Let S be a subset of $\mathbb{Z}^+$. Assume:
   (1) $1 \in S$,
   (2) for all $n \in \mathbb{Z}^+$, if $n \in S$ then also $n + 1 \in S$.
   Then $S = \mathbb{Z}^+$.

   Hint to get you started: If $S \subsetneq \mathbb{Z}^+$, then there is a least positive integer *not* in S.

7. (Laws of exponents) Let $a \in \mathbb{Z}$. Suppose that m, n belong to the set $\mathbb{Z}^+ \cup \{0\}$ of nonnegative integers.

   (a) Prove that $a^m \cdot a^n = a^{m+n}$.

   (b) Prove that $a^{mn} = (a^m)^n$.

   Hint: If $m = 0$ or $n = 0$, this is easy (why?). So you can suppose $m, n \in \mathbb{Z}^+$. Now think of m as fixed and proceed by induction on n.

8. Use the binomial theorem to find formulas for the following sums, as functions of n, where n is assumed to be a natural number.

   (a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$.

   (b) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

9. Show that if $a, b \in \mathbb{Z}^+$ and $a \mid b$, then $a \leq b$.

10. In this exercise we outline a proof of the following statement, which we will be taking for granted in our proof of the Division Theorem: If $a, b \in \mathbb{Z}$ with $b > 0$, the set

$$S = \{a - bq : q \in \mathbb{Z} \text{ and } a - bq \geq 0\}$$

   has a least element.

   (a) Prove the claim in the case $0 \in S$.

   (b) Prove the claim in the case $0 \notin S$ and $a > 0$.

   (c) Prove the claim in the case $0 \notin S$ and $a \leq 0$.

   Hint: (a) is easy. To handle (b) and (c), first show that in these cases that S is a nonempty set of natural numbers, so that the well-ordering principle guarantees S has a least element as long as S is nonempty. To prove S is nonempty, show that in case (b), the integer a is an element of S. You will have to work a little harder to prove S is nonempty in case (c).

11. Use the Euclidean algorithm to find $\gcd(314, 159)$ and $\gcd(272, 1479)$. Show the steps, not just the final answer.

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11. (*) Prove that there is no subset S of the complex numbers $\mathbb{C}$ satisfying all three of the following properties.

    (1) If $a, b \in S$, then $a + b \in S$.

    (2) If $a, b \in S$, then $a \cdot b \in S$.

    (3) For every $a \in \mathbb{C}$, exactly one of the following holds: (a) $a \in S$, (b) $a = 0$, (c) $-a \in S$.

    You may assume familiar properties of $\mathbb{C}$ for this problem.

12. (*) Prove that the properties of the set $\mathbb{Z}^+$ on the handout uniquely determine $\mathbb{Z}^+$ as a set.

    Precisely: Assume all of A1–N1, as usual. Furthermore, assume O1 and WOP hold with two subsets P, P' of the integers in place of $\mathbb{Z}^+$. Prove that $P = P'$.