MATH 4000/6000 – Homework #6
posted April 5, 2024; due April 8, 2024

You can observe a lot by just looking. – Yogi Berra

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

In this assignment, "ring" always means "commutative ring."

1. Let R be a ring. Recall that if $x_1, \dots, x_n$ are elements of R, then (by definition)

$$\langle x_1, \dots, x_n \rangle = \{r_1 x_1 + \dots + r_n x_n : \text{all } r_i \in R\}.$$

   In other words, $\langle x_1, \dots, x_n \rangle$ is the set of all R-linear combinations of $x_1, \dots, x_n$. Prove that $\langle x_1, \dots, x_n \rangle$ is an ideal of R by directly verifying the three defining properties of an ideal.

2. Let R be an integral domain. Show that if $a, b \in R$, then $\langle a \rangle = \langle b \rangle$ if and only if $a = u \cdot b$ for some unit $u \in R$. (Make sure your argument also handles the case when one of a or b is zero.)

3. Let R be a ring in which every ideal is principal. That is, every ideal of R has the form $\langle r \rangle$ for some $r \in R$.

   Let $x_1, \dots, x_n \in R$. Since $\langle x_1, \dots, x_n \rangle$ is an ideal of R, there is some $d \in R$ with $\langle x_1, \dots, x_n \rangle = \langle d \rangle$. Prove that d divides all of $x_1, \dots, x_n$ and that if e is any element of R dividing all of $x_1, \dots, x_n$, then $e \mid d$.

4. Let F be a field. Prove that if I is any ideal of $F[x]$, then $I = \langle f(x) \rangle$ for some $f(x) \in F[x]$. (Imitate the proof from class for the analogous claim in $\mathbb{Z}$.)

5. Let R be a ring, not the zero ring.
   (a) Prove that if $I \subseteq R$ is an ideal and $1 \in I$, then $I = R$.
   (b) Prove that $a \in R$ is a unit if and only if $\langle a \rangle = R$.
   (c) Prove that R is a field if and only if the only ideals in R are $\langle 0 \rangle$ and R.

6. Let F be a field and suppose that $f(x) \in F[x]$ has degree $n \geq 1$. In class, we showed [will show] that the elements of $F[x]/\langle f(x) \rangle$ all have the form $\overline{a_0 + a_1 x + \dots + a_{n-1} x^{n-1}}$, where $a_0, \dots, a_{n-1} \in F$. Show that this representation is unique; that is, distinct choices of a_i lead to distinct elements of $F[x]/\langle f(x) \rangle$.

7. Let F be a field, and suppose $f(x) \in F[x]$ is a nonconstant polynomial that is not irreducible. Show that $F[x]/\langle f(x) \rangle$ is not an integral domain.

   Hint. Think about the multiplication table for $\mathbb{Z}_3[x]/\langle x^2 \rangle$.

8. Write out the addition and multiplication tables for $\mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle$. Is this ring a domain? a field?

9. Let F be a field, and let $f(x) \in F[x]$ be irreducible. Show that $F[x]/\langle f(x) \rangle$ is a field.

   Hint. If $f(x) \nmid a(x)$, then there are $X(x), Y(x) \in F[x]$ with $a(x)X(x) + f(x)Y(x) = 1$. What does this equation tell you in $F[x]/\langle f(x) \rangle$?

10. (*; **MATH 6000 problem**) A polynomial in $\mathbb{Z}[x]$ is called primitive if there is no integer larger than 1 dividing all of its coefficients. For instance, $2x^{10} - 7x + 3$ is primitive, but $3x^{10} - 21x + 3$ is not. Prove that a product of two primitive polynomials is primitive.

    Hint. Find a way to use $\mathbb{Z}_p[x]$ is domain whenever p is prime.

    This result is is the key ingredient in showing Gauss's polynomial lemma.

11. (*; **MATH 6000 problem**) Let $R = \mathbb{Z}[x]$, and let I be the set of elements of R with even constant term. Show that I is an ideal of R but that I is not principal: there is no $f(x) \in \mathbb{Z}[x]$ with $I = f(x)\mathbb{Z}[x]$.