Properties of the integers

$\mathbb{Z}$ is a set with two binary operations, $+$ (addition) and $\cdot$ (multiplication).

Properties of addition

A1. (Existence of an additive identity) There is an element $0 \in \mathbb{Z}$ satisfying $0 + a = a + 0 = a$ for all $a \in \mathbb{Z}$.

A2. (Commutativity of $+$) For all $a, b \in \mathbb{Z}$, we have $a + b = b + a$.

A3. (Associativity of $+$) For all $a, b, c \in \mathbb{Z}$, we have $a + (b + c) = (a + b) + c$.

A4. (Existence of additive inverses) For all $a \in \mathbb{Z}$, there is some $b \in \mathbb{Z}$ with $a + b = 0$.

Properties of multiplication

M1. (Existence of a multiplicative identity) There is an element $1 \in \mathbb{Z}$ satisfying $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{Z}$.

M2. (Commutativity of $\cdot$) For all $a, b \in \mathbb{Z}$, we have $a \cdot b = b \cdot a$.

M3. (Associativity of $\cdot$) For all $a, b, c \in \mathbb{Z}$, we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Distributive law

D1. For all $a, b, c \in \mathbb{Z}$, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (b + c) \cdot a = b \cdot a + c \cdot a.$$

Nontriviality

N1. $0 \neq 1$

$\mathbb{Z}$ is ordered

O1. There is a distinguished subset $\mathbb{Z}^+$ of $\mathbb{Z}$ (the **positive integers**) with the following three properties.

1. If $a, b \in \mathbb{Z}^+$, then $a + b \in \mathbb{Z}^+$.
2. If $a, b \in \mathbb{Z}^+$, then $ab \in \mathbb{Z}^+$.
3. For each $a \in \mathbb{Z}$, **exactly one** of the following holds: $a \in \mathbb{Z}^+$, $a = 0$, or $-a \in \mathbb{Z}^+$.

Using O1, we can define what "$<$" means for integers. Namely, $x < y$ means that $y + (-x) \in \mathbb{Z}^+$, where $-x$ denotes the additive inverse of x. (The word "the" in the last sentence needs some justification, which we will get to.) We define $x \leq y$ to mean that $x < y$ or $x = y$.

Well ordering principle

WOP. Every nonempty subset of $\mathbb{Z}^+$ has a least element. In other words, if $S \subseteq \mathbb{Z}^+$ and $S \neq \emptyset$, then there is an $\ell \in S$ with the property that $\ell \leq x$ for all $x \in S$.