MATH 4000/6000 – Sample Axiom Proofs

I thought it would be helpful to record the kinds of arguments — and level of detail — I am expecting in your solutions to HW 1, Problems #1 and #2. I illustrate by carefully writing up two theorems proved in class, that $0 \cdot a = 0$ and $(-1)a = -a$.

Note that I only include justifications for steps which rely on our axioms. Basic laws of logic (such as the law of identity asserting that $a = a$, or the law that performing the same operation on equal quantities leaves them equal) do **not** require comment.

You are **not** required to use a two column format. However, whatever manner you choose to present your proofs should be *at least as* clear as what is shown below, with axioms (and/or consequences of the axioms derived in class) explicitly referenced when they are used.

Theorem 1. *For every $a \in \mathbb{Z}$,*

$$0 \cdot a = 0.$$

Proof. For each $a \in \mathbb{Z}$,

$$\begin{aligned}
0 &= 0 + 0 && \text{(0 is the additive identity)}\\
a \cdot 0 &= a(0+0) && \\
a \cdot 0 &= a \cdot 0 + a \cdot 0 && \text{(distributive law)}\\
a \cdot 0 + -(a \cdot 0) &= (a \cdot 0 + a \cdot 0) + -(a \cdot 0) && \\
a \cdot 0 + -(a \cdot 0) &= a \cdot 0 + (a \cdot 0 + -(a \cdot 0)) && \text{(associativity of } +\text{)}\\
0 &= a \cdot 0 + 0 && (-a \cdot 0 \text{ is the } + \text{ inv. of } a \cdot 0)\\
0 &= a \cdot 0 && \text{(0 is the additive identity).} \quad \square
\end{aligned}$$

Theorem 2. *For every $a \in \mathbb{Z}$,*

$$(-1) \cdot a = -a.$$

Proof. For each $a \in \mathbb{Z}$,

$$\begin{aligned}
1 + (-1) &= 0 && (-1 \text{ is the } + \text{ inv. of } 1)\\
(1 + (-1))a &= 0 \cdot a && \\
1 \cdot a + (-1) \cdot a &= 0 \cdot a && \text{(distributive law)}\\
1 \cdot a + (-1) \cdot a &= 0 && \text{(Theorem 1, above)}\\
a + (-1) \cdot a &= 0 && (1 \text{ is the } \cdot \text{ identity})\\
-a + (a + (-1) \cdot a) &= -a + 0 && \\
(-a + a) + (-1) \cdot a &= -a + 0 && \text{(associativity of } +\text{)}\\
0 + (-1) \cdot a &= -a + 0 && (a \text{ is the } + \text{ inverse of } -a)\\
(-1) \cdot a &= -a && (0 \text{ is the } + \text{ id.).} \quad \square
\end{aligned}$$