The frequency of partially perfect numbers

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Messing with perfection

Let $s(n) := \sum_{d|n, d<n} d$ denote the sum of the proper divisors of n.
So if $\sigma(n) = \sum_{d|n} d$ is the usual sum-of-divisors function, then

$$s(n) = \sigma(n) - n.$$

For example,

$$s(4) = 1 + 2 = 3, \qquad \sigma(4) = 1 + 2 + 4 = 7.$$

The ancient Greeks said that n was ...
deficient if $s(n) < n$, for instance $n = 5$;
abundant if $s(n) > n$, for instance $n = 12$;
perfect if $s(n) = n$, for example $n = 6$.

Nicomachus (60-120 AD) and the Goldilox theory

The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.

. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.

A deep thought

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A deep thought

We tend to scoff at the beliefs of the ancients.
But we can't scoff at them personally, to their faces, and this is what annoys me.

– Jack Handey

From numerology to number theory

"Serious" problem: How are perfect numbers distributed? More specifically, what can we say about the number of perfect numbers $n \leq x$, as x grows?

Theorem (Euclid)

If $2^m - 1$ *is a prime number, then* $n := 2^{m-1}(2^m - 1)$ *is a perfect number.*

For example, $2^2 - 1$ is prime, so $n = 2 \cdot (2^2 - 1) = 6$ is perfect.

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Theorem (Euler)

If n *is perfect and even, then* n *arises from Euclid's formula.*

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To do better, we need to understand the distribution of m with $2^m - 1$ prime. We know **something** about these m. For example: If $2^m - 1$ is prime, then m itself is prime (HW!). And values of m with $2^m - 1$ prime seem to keep appearing, e.g.,

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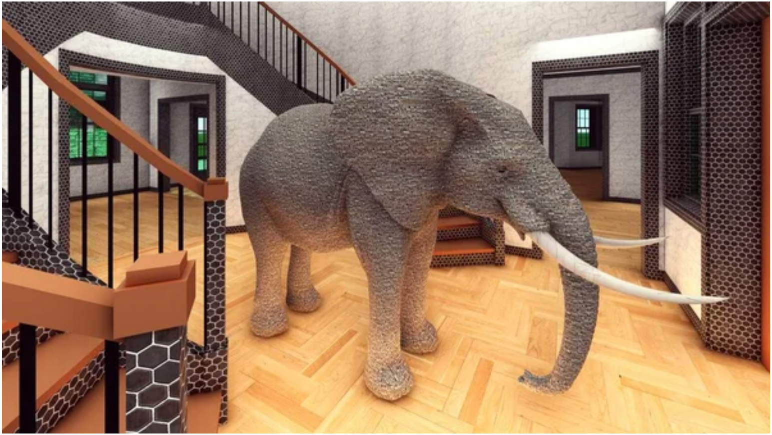
Open problem

Are there infinitely many prime m for which $2^m - 1$ is composite?



What about *odd* perfect numbers?

Conjecture

There are no odd perfect numbers.

Let $V(x)$ denote the number of perfect numbers $n \leq x$. From everything said above, we *expect* that

$$V(x) = O(\log x).$$

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What can we prove?

Theorem (Hornfeck–Wirsing, 1957)

For each $\epsilon > 0$*, and all* $x > x_0(\epsilon)$*,*

$$V(x) < x^{\epsilon}.$$

Wirsing, 1959: $V(x) \leq x^{c/\log\log x}$.

Partially perfect?

If n is perfect, then

$$\gcd(n, s(n)) = \gcd(n, n) = n.$$

Since $\gcd(n, \cdot) \leq n$, one could say perfect numbers are example of maximizers for the function $\gcd(n, s(n))$.

The maximizers are precisely the n dividing $s(n)$: such n are called **multiperfect**. We understand the distribution of multiply perfect numbers at about the same level as we do perfect numbers. We don't know how to prove there are infinitely many. But the theorems of Hornfeck–Wirsing and Wirsing from the last slide still apply.

If one is thinking in this way, it's natural to ask how often $\gcd(n, s(n))$ is as *small* as possible. That is, how many $n \leq x$ have $\gcd(n, s(n)) = 1$. This was answered by Erdős in 1948: *As* $x \to \infty$,

$$\frac{1}{x}\#\{n \leq x : \gcd(n, s(n)) = 1\} \sim \frac{e^{-\gamma}}{\log\log\log x}.$$

(Here γ is the Euler–Mascheroni constant that appears when estimatings partial sums of the harmonic series.)

The denominator $\log\log\log x$ tends to infinity here, but very slowly. So while the limiting frequency of n with $\gcd(n, s(n)) = 1$ is 0%, it is a pretty "fat" 0% !

Large, small, and everything inbetween

Define the two-variable function

$$E(x, y) = \#\{n \leq x : \gcd(n, s(n)) > y\}.$$

Question

What estimates can we prove for $E(x, y)$?

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Question

What estimates can we prove for $E(x, y)$?

Notice that if $n \in (x/2, x]$ and n is multiperfect, then $\gcd(n, s(n)) = n \geq x/2$, so n is counted by $E(x, x/2)$. Hence, the count of multiply perfect $n \leq x$ is at most

$$E(x, x/2) + E(x/2, x/4) + E(x/4, x/8) + \dots.$$

So upper bounds for $E(x, y)$ yield upper bounds for counts of multiperfects.

Theorem

Fix $\epsilon > 0$. *If* $x, y \to \infty$ *with* $x^\epsilon \le y \le x^{1-\epsilon}$, *then*

$$E(x, y) = x/y^{1+o(1)}.$$

Example: There are $x^{2/3+o(1)}$ values of $n \le x$ with $\gcd(n, s(n)) > x^{1/3}$.

Consequently: There are $O(x^\epsilon)$ multiply perfect $n \le x$.

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Consequently: There are $O(x^\epsilon)$ multiply perfect $n \leq x$.

Actually, this consequence is a bit of a cheat: The proof uses Hornfeck–Wirsing. In fact, the theorem uses a more general form of the Hornfeck–Wirsing result where one counts solutions not only to $s(n) = \alpha n$, for given rational α.

Theorem

Fix $\epsilon > 0$. If $x, y \to \infty$ with $x^{\epsilon} \le y \le x^{1-\epsilon}$, then

$$E(x, y) = x/y^{1+o(1)}.$$

The proof shows that lower bound holds when y grows faster than any fixed power of $x^{1/\sqrt{\log\log x}}$.

When is $x/y^{1+o(1)}$ the correct answer? We don't know!

Easier question: When do we have an upper bound saving *some* power of y? That is: In what range do we have $E(x, y) \le x/y^{\delta}$, with $\delta > 0$ bounded away from 0? It's "easy" to prove that this fails if y is too small.

Erdős's claim

It is necessary and sufficient that $y > (\log x)^{\beta}$*, with* $\beta > 0$ *bounded away from* 0*.*

Erdős's claim (corrected)

It is necessary and sufficient that $y > \exp((\log\log x)^{\beta})$*.*

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Erdős's claim (corrected)

It is necessary and sufficient that $y > \exp((\log\log x)^{\beta})$.

The results from this summer probe the behavior of $E(x, y)$ when $y = \exp((\log\log x)^{v})$ with small v. Write $y = (\log\log x)^{u}$. If $u \to \infty$ but $v \to 0$, then

$$E(x, y) = x \exp(-(1 + o(1))u \log u).$$

Consequently, if v goes to 0 but not too quickly,

$$E(x, y) = x/y^{v(1+o(1))}.$$


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