DENSITIES OF INTEGER SETS REPRESENTED BY QUADRATIC FORMS

PETE L. CLARK, PAUL POLLACK, JEREMY ROUSE, AND KATHERINE THOMPSON

Abstract. Let $f(t_1,\ldots,t_n)$ be a nondegenerate integral quadratic form. We analyze the asymptotic behavior of the function $D_f(X)$, the number of integers of absolute value up to X represented by f. When f is isotropic or n is at least 3, we show that there is a $\delta(f) \in \mathbb{Q} \cap (0,1)$ such that $D_f(X) \sim \delta(f)X$ and call $\delta(f)$ the **density** of f. We consider the inverse problem of which densities arise. Our main technical tool is a Near Hasse Principle: a quadratic form may fail to represent infinitely many integers that it locally represents, but this set of exceptions has density 0 within the set of locally represented integers.

1. Introduction

1.1. **Terminological preliminaries.** Let S be a subset of $\mathbb{Z}$. We say that S is **positive** if it contains no negative integers, **negative** if it contains no positive integers, **definite** if it is either positive or negative and **indefinite** otherwise.

We say a subset $S \subset \mathbb{Z}$ has **density** $\delta = \delta(S)$ if

$$\begin{cases} \lim_{N\to\infty} \frac{\# S\cap[1,N]}{N} = \delta & \text{if } S \text{ is positive,}\\ \lim_{N\to\infty} \frac{\# S\cap[-N,-1]}{N} = \delta & \text{if } S \text{ is negative,}\\ \lim_{N\to\infty} \frac{\# S\cap[-N,N]}{2N} = \delta & \text{if } S \text{ is indefinite;}\end{cases}$$

in each case we are requiring the limit to exist. We say that $S \subset \mathbb{Z}$ has **upper density** $\overline{\delta}$ (resp. **lower density** $\underline{\delta}$) if in the above definition we replace $\lim_{N\to\infty}$ by $\limsup_{N\to\infty}$ (resp. by $\liminf_{N\to\infty}$).

Let R be a PID of characteristic different from 2 with fraction field K. We denote by $R^\bullet$ the set of nonzero elements of R. For a prime element p of R, let v_p be the corresponding p-adic valuation. Let $f \in R[t_1,\ldots,t_n]$ be a nondegenerate quadratic form. (Henceforth we assume all quadratic forms to be nondegenerate.) We put

$$D_f \coloneqq f(R^n) \setminus \{0\}.$$

A quadratic form f is **primitive** if its coefficients generate the unit ideal of R. The quadratic form f corresponds to a R-lattice (necessarily free, since R is a PID) L in a quadratic space (V,q) over K on which f is R-valued. We say that q is **maximal** if there is no R-lattice $M \supsetneq L$ on which q is R-valued.

We say that a quadratic form $f_{/R}$ is **ADC** (cf. [Cl12], [CJ14]) if for all $x \in R$, whenever there is $v \in K^n$ such that $f(v) = x$, there is $w \in R^n$ such that $f(w) = x$. In other words, f is ADC if and only if every element of R that is K-represented by f is moreover R-represented by f. In symbols:

$$D_f = D_{f_{/K}} \cap R.$$

For a prime element p of R, let K_p be the completion of K with respect to the p-adic valuation v_p, and let R_p be the valuation ring. We say that $f_{/R}$ is **locally ADC** if for all prime elements p, the form $f_{/R_p}$ is ADC.

A quadratic form $f_{/\mathbb{Z}}$ is **regular** if for all $m \in \mathbb{Z}$, if f $\mathbb{R}$-represents m and $\mathbb{Z}_p$-represents m for all primes p, then f $\mathbb{Z}$-represents m. A quadratic form $f_{/\mathbb{Z}}$ is **almost regular** if the set

$$\{m \in \mathbb{Z} \mid f \text{ represents } m \text{ over } \mathbb{R} \text{ and over } \mathbb{Z}_p \text{ for all primes } p \text{ but not over } \mathbb{Z}\}$$

is finite. (In this paper, the term "almost" in this paper will always mean "all but finitely many.")

Theorem 1.1.

a) *Let $f_{/R_p}$ be a quadratic form. If f is maximal, then it is ADC.*
b) *Let $f_{/R}$ be a quadratic form. If f is ADC, then it is locally ADC.*
c) *Let $f_{/\mathbb{Z}}$ be a quadratic form. Then f is ADC if and only if it is locally ADC and regular.*

Proof. These are either all results from [Cl12] or follow immediately from them.
a) Combine [Cl12, Thm. 8] and [Cl12, Thm. 19].
b) This is [Cl12, Cor. 17].
c) This is [Cl12, Thm. 25]. $\square$

1.2. **The density of an integral quadratic form.** Let $f \in \mathbb{Z}[t_1, \ldots, t_n]$ be an n-ary integral quadratic form. As a special case of a definition given above, we put

$$D_f := f(\mathbb{Z}^n) \setminus \{0\}.$$

Moreover, for $X \geq 1$, let

$$D_f(X) := D_f \cap [-X, X].$$

We call f positive, negative or indefinite according to whether D_f is positive, negative or indefinite. If f is negative, then $-f$ is positive and $D_f(X) = -D_{-f}(X)$, so negative forms do not merit separate consideration.

One of the main problems in the area is, given f, to determine D_f explicitly. In general this is quite difficult. Actually the situation is worse: it is not completely clear what "determine D_f explicitly" means! The set D_f is recursive: there is an algorithm that, given f and $m \in \mathbb{Z}$, determines whether f represents m. Presumably we have in mind some *finitistic* description of D_f. This is possible e.g. if f is known to be almost regular, but even so there are 14 positive ternary forms that are known to be regular conditionally on GRH but not yet unconditionally [LO14].

1.3. **Some motivating examples.** The point of departure of this work is the idea that it ought to be simpler to describe the *size* of D_f rather than D_f itself: namely in terms of the density

$$\delta(f) := \delta(D_f)$$

and – when $\delta(f) = 0$ – the asymptotic behavior of $D_f(X)$. We consider several motivating examples.

Example 1.2.

a) Let $f = t_1 t_2$, an indefinite form. Then $D_f = \mathbb{Z} \setminus \{0\}$, so $\delta(f) = 1$.
b) Let $f = t_1^2 - t_2^2$, an indefinite form. Then

$$D_f = \{m \in \mathbb{Z}^\bullet \mid v_2(m) \neq 1\},$$

so $\delta(f) = \frac{3}{4}$.

Example 1.3 (Fermat)**.** Let $f = t_1^2 + t_2^2$, a positive form. As Fermat knew, f represents $m \in \mathbb{Z}^+$ if and only if $v_p(m)$ is even for all primes $p \equiv 3 \pmod 4$. Thus for all primes $p \equiv 3 \pmod 4$, f does not represent any integer that is divisible by p but not by p^2. Since

$$\prod_{p \equiv 3 \pmod 4} \left(1 - \frac{1}{p} + \frac{1}{p^2}\right) = 0,$$

it follows that $\delta(f) = 0$.

Example 1.4 (Gauss-Legendre)**.** Let $f = t_1^2 + t_2^2 + t_3^2$, a positive form. Gauss and Legendre showed that f represents $n \in \mathbb{Z}^+$ if and only if n is *not* of the form $4^a(8b + 7)$ for $a, b \in \mathbb{N}$. It follows that

$$\delta(f) = 1 - \frac{1 + 1/4 + 1/4^2 + \ldots}{8} = \frac{5}{6}.$$

Example 1.5.

a) Let $f = t_1^2 + t_2^2 + t_3^2 + t_4^2$. Lagrange showed that $D_f = \mathbb{Z}^+$, so $\delta(f) = 1$.
b) Let $f = 2023t_1^2 + 2023t_2^2 + 2023t_3^2 + 2023t_4^2$. Then $D_f = 2023\mathbb{Z}^+$ so $\delta(f) = \frac{1}{2023}$.
c) For a prime number p, let $f_p = t_1^2 + p^2t_2^2 + \ldots + p^2t_4^2$. Then f_p does not represent any integer with p-adic valuation 1, so

$$\delta(f_p) \leq 1 - \frac{1}{p} + \frac{1}{p^2} < 1.$$

1.4. **Results on density.** We observe a stark difference in behavior between Examples 1.2 and 1.3. This is due to the fact the form $t_1^2 + t_2^2$ is positive whereas the forms t_1t_2 and $t_1^2 - t_2^2$ are not just indefinite but isotropic. More generally, let $f_{/\mathbb{Z}} = At_1^2 + Bt_1t_2 + Ct_2^2$ be a binary quadratic form of Discriminant[1] $\Delta := B^2 - 4AC \neq 0$. Then: f is isotropic if and only if $\Delta = b^2$ for some $b \in \mathbb{Z}^+$. (This follows, for instance, from [L, Thm. I.3.2].)

Theorem 1.6. *Let $f_{/\mathbb{Z}}$ be a primitive binary quadratic form of Discriminant Δ.*

a) (Landau-Bernays [La08], [Be12]) *Suppose f is anisotropic. There is a constant $\kappa > 0$ (depending only on Δ) such that as $X \to \infty$ we have*

$$\#D_f(X) \sim \frac{\kappa X}{\log^{1/2} X}.$$

b) *If f is isotropic of Discriminant $\Delta = b^2$, then $\delta(f) > 0$. More precisely, for a prime p, let $a_p = v_p(\Delta)$. Put*

$$\delta_2(f) := \begin{cases} 1 & \text{if } a_2 = 0, \\ \frac{3}{4} & \text{if } a_2 = 1, \\ \frac{2+2^{4-a_2}+2^{5-a_2}+2^{2-a_2}}{12} & \text{if } a_2 \geq 2, \end{cases}$$

and for $p > 2$, put

$$\delta_p(f) := \frac{p + p^{1-a_p} + 2p^{-a_p}}{2p+2}.$$

Then

$$\delta(f) = \prod_p \delta_p(f), \tag{1}$$

the product extending over all prime numbers – or equivalently, over all primes dividing Δ: we have $\delta_p(f) = 1$ if and only if $a_p = 0$.

Theorem 1.7. *Let $n \geq 3$, and let $f_{/\mathbb{Z}}$ be an n-ary quadratic form. Then $\delta(f) > 0$.*

Theorem 1.8. *Let $f_{/\mathbb{Z}}$ be an anisotropic ternary quadratic form. Then $\delta(f) < 1$.*

Remark 1.9. It is immediate that for all $a \in \mathbb{Z} \setminus \{0\}$, we have $\delta(ax^2) = 0$. So it follows from Theorems 1.6, 1.7 and 1.8 that the density $\delta(f)$ exists for any quadratic form $f_{/\mathbb{Z}}$.

Remark 1.10. Theorem 1.8 is essentially classical. It follows from the Hasse-Minkowski theory that f is anisotropic at some prime number p and that f fails to $\mathbb{Q}_p$-represent $-\operatorname{disc} f$, and thus f fails to represent all integers in a nonempty union of congruence classes modulo p^4 (if $p > 2$ we may replace p^4 by p^2). An extensive analysis of such local representation issues will be given later on.

In particular, no positive definite ternary form is universal, a result that goes back at least to 1933 [Al33]. See [MO] and [DW17] for further information on the history. The latter work gives an explicit congruence class of integers not represented by any given definite ternary f using elementary methods.

Theorem 1.11. *Let $n \geq 3$, and let $f_{/\mathbb{Z}}$ be a quadratic form. If $n = 3$ we assume that f is isotropic.*

a) *The following are equivalent:*
   (i) *f is locally universal: $f_{/\mathbb{Z}_p}$ is universal for all primes p.*

[1] This Discriminant (note the capitalization) is defined only for binary quadratic forms. On the other hand, for an n-ary quadratic form q defined over a domain of characteristic different from 2, we define the discriminant disc(f) to be the determinant of the Gram matrix of the associated bilinear form.

(ii) *f is locally ADC: $f_{/\mathbb{Z}_p}$ is ADC for all primes p.*
(iii) *We have $\delta(f) = 1$.*

b) *If f is maximal, then $\delta(f) = 1$.*

These results invite us to consider the "inverse problem" for densities of representation sets: which real numbers arise as the density $\delta(f)$ of an n-ary quadratic form? Here are some results on this:

Theorem 1.12 (Rationality of densities). *For $f_{/\mathbb{Z}}$ a quadratic form, we have $\delta(f) \in \mathbb{Q}$.*

Theorem 1.13 (Density of densities). *Let $n \geq 3$, and let $r, s \in \mathbb{N}$ be such that $r + s = n$. Let $\mathcal{S}_{r,s}$ be the set of n-ary quadratic forms $f_{/\mathbb{Z}}$ with signature (r, s). Then the set*

$$\mathcal{D}_{r,s} := \{\delta(f) \mid f \in \mathcal{S}_{r,s}\}$$

is dense in $[0, 1]$.

Theorem 1.14. *As $f_{/\mathbb{Z}}$ varies over all locally ADC ternary quadratic forms, the possible values of $\delta(f)$ comprise only 0% of all rational numbers in $[0, 1]$, in the sense of height. Since maximal quadratic forms are locally ADC, the same holds as we vary over all maximal ternary $f_{/\mathbb{Z}}$.*

We do not know whether for every $\delta \in (0, 1) \cap \mathbb{Q}$ there is a ternary quadratic form $f_{/\mathbb{Z}}$ with $\delta(f) = \delta$.

1.5. **The Near Regularity Theorem and the Density Hasse Principle.** The key to the proofs of the above results is a Hasse principle for $\delta(f)$. For a quadratic form $f_{/\mathbb{Z}}$, we say that that $m \in \mathbb{Z}$ is **locally represented** by f if f represents m over $\mathbb{Z}_p$ for all primes p and also over $\mathbb{R}$. Let $D_{f,\mathrm{loc}}$ be the set of integers n that are locally represented by f. Thus we have $D_f \subset D_{f,\mathrm{loc}}$, with equality if and only if f is regular, whereas almost regularity means that $D_f \setminus D_{f,\mathrm{loc}}$ is finite.

Theorem 1.15 (Tartakowsky-Kloosterman-Ross-Pall [RP46]). *Let $f_{/\mathbb{Z}}$ be a positive quadratic form. If $f_{/\mathbb{Z}_p}$ is isotropic for all primes p, then f is almost regular.*

When $n \geq 5$, every n-ary quadratic form over $\mathbb{Z}_p$ is isotropic, and thus we get:

Corollary 1.16 (Tartakowsky). *For $n \geq 5$, every n-ary positive $f_{/\mathbb{Z}}$ is almost regular.*

Remark 1.17. Combining Corollary 1.16 and Theorem 1.1a) gives: if $n \geq 5$, a maximal positive n-ary $f_{/\mathbb{Z}}$ represents all sufficiently large integers.

There are also positive quaternary forms $f_{/\mathbb{Z}}$ such that $f_{/\mathbb{Z}_p}$ is isotropic for all primes p, e.g. $t_1^2 + t_2^2 + t_3^2 + nt_4^2$ where $n \in \mathbb{Z}^+$ is not of the form $4^a(8b + 1)$.

Theorem 1.18 (Bochnak-Oh [BO08]). *Let $f_{/\mathbb{Z}}$ be an almost regular positive quaternary form. Then the set of anisotropic primes of f is either empty or consists of a single prime $p \leq 37$.*

Example 1.19.

a) The binary quadratic form $t_1^2 + 14t_2^2$ is *not* almost regular. The set of prime numbers that it represents locally has relative density $\frac{1}{4}$, whereas the set of prime numbers that it represents has relative density $\frac{1}{8}$.
b) The ternary quadratic form $3t_1^2 + 4t_2^2 + 9t_3^2$ is *not* almost regular (more details are given in Example 4.5).
c) The quadratic form $f = t_1^2 + t_2^2 + 7t_3^2 + 7t_4^2$ locally represents all positive integers. Since $f \equiv 0 \pmod{49}$ if and only if $t_1 \equiv t_2 \equiv t_3 \equiv t_4 \equiv 0 \pmod 7$, then f is anisotropic at 7. The fact that f does not represent 3, 6, 21 or 42 shows that f fails to represent $3 \cdot 7^k$ or $6 \cdot 7^k$ for any integer $k \geq 0$. This shows that f is *not* almost regular (as was known to Watson [Wa, p. 121]). More effort shows that f represents every positive integer except those of the form $3 \cdot 7^k$ or $6 \cdot 7^k$.

So positive forms in fewer than five variables need not be almost regular. However, there is a more permissive – but analytically natural – sense in which every quadratic form is "nearly regular." Put

$$D_{f,\mathrm{loc}}(X) := D_{f,\mathrm{loc}} \cap [-X, X]$$

and

$$\mathcal{E}_f(X) \coloneqq D_{f,\mathrm{loc}}(X) \setminus D_f(X).$$

Let us also put

$$\delta_{\mathrm{loc}}(f) \coloneqq \delta(D_{f,\mathrm{loc}}).$$

For a prime number p, let μ_p be the unique Haar measure on $\mathbb{Z}_p$ with $\mu_p(\mathbb{Z}_p) = 1$. For an n-ary integral quadratic form f, we define $\delta_p(f)$ to be $\mu_p(\{f(x_1,\ldots,x_n) \mid (x_1,\ldots,x_n) \in \mathbb{Z}_p^n\})$. (These are the quantities $\delta_p(f)$ that appear in Theorem 1.6.)

We have the following two key results.

Theorem 1.20 (Product Formula)**.** *Let $f_{/\mathbb{Z}}$ be an n-ary quadratic form. Then:*

$$\delta_{\mathrm{loc}}(f) = \prod_p \delta_p(f), \tag{2}$$

the product extending over all prime numbers.

We will see later that if $n \leq 2$ and f is anisotropic, then $\prod_p \delta_p(f) = 0$. In every other case we have $\delta_p(f) = 1$ for all sufficiently large p, so the product is actually finite. Thus Theorem 1.20 implies that $\delta_{\mathrm{loc}}(f)$ exists.

Theorem 1.21 (Near Regularity)**.** *Let $f_{/\mathbb{Z}}$ be an n-ary quadratic form.*

a) *As $X \to \infty$,*

$$\#\mathcal{E}_f(X) = o(D_f(X)). \tag{3}$$

b) *If $n \geq 3$ we have $\#\mathcal{E}_f(X) = O(\sqrt{X})$ (whereas Theorem 1.7 gives $D_f(X) \gg X$).*

These results have the following immediate consequence:

Corollary 1.22 (Density Hasse Principle)**.** *For all $f_{/\mathbb{Z}}$, we have*

$$\delta(f) = \delta_{\mathrm{loc}}(f) = \prod_p \delta_p(f).$$

Corollary 1.22 shifts the work of computing $\delta(f)$ to the more tractable setting of quadratic forms over $\mathbb{Z}_p$. In fact we carry out this local analysis first in §2. In §3 we prove Theorem 1.20. We prove Theorem 1.21 in §4, in fact with an explicit error bound that depends on the number of variables and whether f is definite or indefinite. In §5 we prove a globalization result, which is used in §6 along with Corollary 1.22 to prove the remaining results.

2. The local case

2.1. **The representation table and the local representation measure.** In this paper, a **local field** is a complete discretely valued field (K, v) of characteristic different from 2, with valuation ring R, uniformizing element π, and finite residue field $R/\pi R \cong \mathbb{F}_q$. We say that K (or R) is **nondyadic** if q is odd and **dyadic** if q is even. Let $\mathcal{S} \coloneqq K^\times/K^{\times 2}$ be the set of square classes of K. By a slight abuse of notation, we will also denote by $\mathcal{S}$ a chosen set of representatives for $K^{\times 2}$ in $K^\times$.

If R is non-dyadic, we fix $r \in R^\times$ such that $r + \pi R \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^{\times 2}$. Then as coset representatives for $K^{\times 2}$ in K we may take [L, Thm. VI.2.2(1)]

$$\mathcal{S} = \{1, r, \pi, r\pi\}.$$

If R is dyadic then $\#\mathcal{S} = 2^{[K:\mathbb{Q}_2]+2}$ [L, Cor. VI.2.23]. When $K = \mathbb{Q}_2$, as coset representatives for $\mathbb{Q}_2^{\times 2}$ in $\mathbb{Q}_2^\times$ we take

$$\mathcal{S} = \{1, 3, 5, 7, 2, 6, 10, 14\}.$$

For $n \geq 1$, let $f_{/R}$ be an n-ary quadratic form. In contrast to the global case, we can give a finitistic description of D_f in all cases. Namely, for $s \in \mathcal{S}$, let v_s be the minimal valuation $v(x)$ of an element $x \in D_f$ such that $x \in sK^{\times 2}$, or ∞ if no such element exists. We refer to the assignment

$$s \in \mathcal{S} \mapsto v_s \in \mathbb{N} \cup \{\infty\}$$

as the **representation table** of f. Let $s \in \mathcal{S}$ with $v_s < \infty$. Then there is $x \in D_f \cap sK^{\times 2}$ with valuation v_s; moreover, for any element $y \in sK^{\times 2}$ with $v(y) = v_s$ we have $y = u^2 x$ for some $u \in R^\times$, so $y \in D_f$. Moreover, since $x \in D_f$, so is $\pi^{2k}x$ for all $k \geq 0$. It follows that

$$D_f \cap sK^{\times 2} = \{y \in R^\bullet \mid y \in sK^{\times 2} \text{ and } v(y) \geq v_s\},$$

and thus knowing the representation table determines D_f.

Remark 2.1. Let $f_{/R}$ be an n-ary quadratic form. Here is some information on the finiteness / infiniteness of the entries in the representation table of f.

a) Let $s \in \mathcal{S}$. Then $v_s < \infty$ if and only if f K-represents s.
b) If f is isotropic, then $f_{/K}$ is K-universal, so $v_s < \infty$ for all $s \in \mathcal{S}$.
c) For a unary form we have $v_s < \infty$ for a unique $s \in \mathcal{S}$.
d) If f is an anisotropic binary form, then $f_{/K}$ K-represents precisely half of the elements of $\mathcal{S}$ [L, p. 184, Exc. 8], so precisely half of the entries of the representation table of f will be finite. If K is nondyadic, we will shortly see by direct calculation that all 6 possible pairs of square classes can arise this way. On the other hand, when $K = \mathbb{Q}_2$, the inequality $\binom{8}{4} > 8^2$ shows that not all 4 element subsets of $\mathcal{S}$ arise as the set of square classes represented by an anisotropic binary quadratic form $f_{/\mathbb{Q}_2}$. A straightforward asymptotic analysis shows as $a := [K : \mathbb{Q}_2]$ tends to infinity, the number of 2^{a+1} element subsets of $\mathcal{S}$ arising as the set of square classes K-represented by an anisotropic binary quadratic form $f_{/K}$ divided by the number $\binom{2^{a+2}}{2^{a+1}}$ of 2^{a+1} element subsets of $\mathcal{S}$ approaches 0.
e) If f is an anisotropic ternary form, then $f_{/K}$ K-represents every $s \in \mathcal{S}$ except the class of $-\operatorname{disc}(f)$ and does not represent $-\operatorname{disc}(f)$ [L, Cor. 2.15(2)]. So in this case exactly one entry of the representation table will be infinite, and by scaling any one anisotropic ternary form we see that this infinite entry can be any $s \in \mathcal{S}$.
f) If $n \geq 4$, then $f_{/K}$ is universal [L, Cor. 2.11 and Cor. 2.15(1)], so every entry in the representation table is finite. Moreover, if $n \geq 5$ then f is isotropic [L, Thm. 2.12].

Let μ be the Haar measure on R with unit mass. Then we define the **local representation measure**

$$\delta_v(f) := \mu(D_f).$$

When $K = \mathbb{Q}_p$ we write $\delta_p(f)$ instead. We also define

$$\nu(R) := [R^\times : R^{\times 2}],$$

so $\nu(R) = 2$ if q is odd and $\nu(R) = 2^{[K:\mathbb{Q}_2]+1}$ is q is even.

Let $a \in R^\bullet$ have valuation v. If $u := \frac{a}{\pi^v}$ then we get

$$aR^{\times 2} = \pi^v(uR^{\times 2}).$$

For any measurable $Y \subset R$ and $x \in R^\bullet$, we have $\mu(xY) = \frac{\mu(Y)}{q^{v(x)}}$. It follows that

$$\mu(R^\times) = \mu(R \setminus \pi R) = \frac{q-1}{q},$$

$$\mu(uR^{\times 2}) = \frac{q-1}{\nu(R)q}$$

and finally that

$$\mu(aR^{\times 2}) = \frac{q-1}{\nu(R)q^{v+1}}.$$

Let $s \in \mathcal{S}$ be such that $v_s < \infty$, and choose $x \in D_f \cap sK^{\times 2}$ with $v(x) = v_x$. Then

$$\mu(D_f \cap sK^{\times 2}) = \mu\left(\coprod_{i=0}^{\infty} \pi^{2i} x R^{\times 2}\right)$$
$$= \mu(xR^{\times 2})\left(1 + \frac{1}{q^2} + \frac{1}{q^4} + \ldots\right) = \frac{q^2}{q^2-1}\mu(xR^{\times 2}) = \frac{1}{\nu(R) q^{v_s - 1}(q+1)}.$$

When $v_s = \infty$, we take the expression $\frac{1}{\nu(R)q^{v_s-1}(q+1)}$ to be 0. This yields a formula for the local representation measure $\delta_v(f)$ in terms of the representation table:

Theorem 2.2. *Let $n \in \mathbb{Z}^+$, and let $f_{/R}$ be an n-ary quadratic form. Then we have*

$$\delta_v(f) = \sum_{s \in \mathcal{S}} \frac{1}{\nu(R) q^{v_s - 1}(q+1)}. \tag{4}$$

Here is an immediate consequence:

Corollary 2.3. *Let $n \in \mathbb{Z}^+$, and let $f_{/R}$ be an n-ary quadratic form. Then:*

a) *We have $\delta_v(f) \in (0,1] \cap \mathbb{Q}$.*
b) *We have $D(f) = R^\bullet$ if and only if $\delta_v(f) = 1$.*

If $f_{/R}$ is a quadratic form, it is a routine task to compute its representation table and thus via Theorem 2.2 its local representation measure $\delta_v(f)$. However, for our later applications we want to solve a related *inverse problem*: namely, for fixed $n \in \mathbb{Z}^+$ we would like to determine all possible representation tables for n-ary quadratic forms $f_{/R}$. (What we actually need is the set of local representation measures, which we approach via the representation tables.) For fixed R and n, this is clearly a finite problem. It is even a finite problem for fixed R and varying n: one can show that the set of representation tables for n-ary forms $f_{/R}$ is independent of n as long as n is sufficiently large.

Here we will solve this problem for every *nondyadic* R and show that the set of possible representation tables of n-ary forms is the same for all $n \geq 4$. To do so we need only hand calculations.

For our applications to integral forms it would be desirable also to treat the case of $R = \mathbb{Z}_2$. But this finite problem seems several orders of magnitude more difficult than the general nondyadic case. Instead we will compute all possible local representation densities $\delta_2(f)$ for (i) isotropic binary forms $f_{/\mathbb{Z}_2}$ and (ii) ADC forms $f_{/\mathbb{Z}_2}$.

2.2. **Non-dyadics.** Throughout this section we suppose that R is non-dyadic, i.e., $\mathbb{F}_q = R/\pi R$ has characteristic different from 2. We order the square classes as $1, r, \pi, r\pi$. If $f_{/R}$ is a quadratic form with representation table $(\alpha, \beta, \gamma, \delta)$, then α, β, if finite, are even, while γ, δ (if finite), are odd.

Since R is non-dyadic, every quadratic form $f_{/R}$ is diagonalizable [Ge, Thm. 8.1], so we may assume

$$f = a_1 t_1^2 + \ldots + a_n t_n^2$$

with $v(a_1) \leq \ldots \leq v(a_n)$. Then $\frac{1}{a_1} f$ is a primitive quadratic form defined over R.

Remark 2.4. The representation table for f is easily determined from that of $\frac{1}{a_1} f$: indeed, ordering the square classes as $1, r, \pi, r\pi$, if the representation table for $\frac{1}{a_1} f$ is $(\alpha, \beta, \gamma, \delta)$, then $a_1 = r^\epsilon u^2 \pi^k$ with $\epsilon \in \{0, 1\}$, $u \in R^\times$ and $k \geq 0$. Then:

- If $\epsilon = 0$ and k is even, the table for f is $(\alpha + k, \beta + k, \gamma + k, \delta + k)$.
- If $\epsilon = 0$ and k is odd, the table for f is $(\gamma + k, \delta + k, \alpha + k, \beta + k)$.
- If $\epsilon = 1$ and k is even, the table for f is $(\beta + k, \alpha + k, \delta + k, \gamma + k)$.
- If $\epsilon = 1$ and k is odd, the table for f is $(\delta + k, \gamma + k, \beta + k, \alpha + k)$.

Thus we have reduced to the case of

$$f = t_1^2 + a_2 t_2^2 + \ldots + a_n t_n^2.$$

2.2.1. *Binary forms.* The discriminant of $t_1^2 + a_2 t_2^2$ is a_2, so as a_2 ranges over all 4 elements of $\mathcal{S}$ we get four different K-isomorphism classes of binary forms.

Case 2.1: $f = t_1^2 - \pi^{2b} t_2^2$. We claim that the representation table is $(0, 2b, 2b+1, 2b+1)$.

The form f is isotropic and thus K-universal. Moreover, taking $t_1 \mapsto \pi^b t_1$, we find that f represents $\pi^{2b}(t_1^2 - t_2^2)$. Since $2 \in R^\times$, $t_1^2 - t_2^2$ is isomorphic to the hyperbolic plane $t_1 t_2$ and thus is universal. It follows that f represents every element with valuation at least $2b$.

• Suppose $x_1^2 - \pi^{2b} x_2^2 = r\pi^{2b-2}$. Then we must have $x_1 = \pi^{b-1} X_1$, so

$$X_1^2 - \pi^2 x_2^2 = r,$$

and going mod π gives $X_1^2 \equiv r \pmod{\pi}$, a contradiction.

• Suppose $x_1^2 - \pi^{2b} z_2^2 = u\pi^{2b-1}$, where $u \in \{1, r\}$. Then $2v(x_1) = v(x_1^2) \geq 2b - 1$ and thus the left hand side has valuation at least $2b$, contradiction.

Case 2.2: $f = t_1^2 - r\pi^{2b} t_2^2$. We claim that the representation table is $(0, 2b, \infty, \infty)$.

The form f is anisotropic and unimodular and thus K-represents 1 and r but not π and $r\pi$. Moreover f represents $\pi^{2b}(t_1^2 - rt_2^2)$. The form $t_1^2 - rt_2^2$ R-represents both 1 and r, by Hensel's Lemma. So f represents $r\pi^{2b}$.

• Suppose $x_1^2 - r\pi^{2b} x_2^2 = r\pi^{2b-2}$. Then $x_1 = \pi^{b-1} X_1$, so

$$X_1^2 - r\pi^2 x_2^2 = r,$$

and going mod π gives a contradiction.

Case 2.3: $f = t_1^2 + \pi^{2b+1} t_2^2$. Then f is anisotropic and K-represents 1 and π but not r or $r\pi$. We claim that the representation table is $(0, \infty, 2b+1, \infty)$. Clearly f represents π^{2b+1}.

• Suppose $x_1^2 + \pi^{2b+1} x_2^2 = \pi^{2b-1}$. Then as above π^{2b} divides the left hand side, contradiction.

Case 2.4: $f = t_1^2 + r\pi^{2b+1} t_2^2$. Then f is anisotropic and K-represents 1 and $r\pi$ but not r or π. The representation table is $(0, \infty, \infty, 2b+1)$; the computations are similar to those of Case 2.3.

2.2.2. *Ternary Forms.*

Case 3.1: $f = t_1^2 - \pi^{2b} t_2^2 + ct_3^2$ with $v(c) \geq 2b$.
We claim that the representation table is $(0, 2b, 2b+1, 2b+1)$, i.e., the same as for its binary subform $t_1^2 - \pi^{2b} t_2^2$, treated in Case 2.1. All we need to show is that for $d = r\pi^{2b-2}$, π^{2b-1}, $r\pi^{2b-1}$, the equation

$$x_1^2 - \pi^{2b} x_2^2 + cx_3^2 = d$$

is not solvable for $(x_1, x_2, x_3) \in R^3$. The argument of Case 2.1 carries over almost verbatim.

Case 3.2.1: $f = t_1^2 - r\pi^{2b} t_2^2 - \pi^{2c} t_3^2$ with $b \leq c$.
We claim that the representation table is $(0, 2b, 2c+1, 2c+1)$. Since f represents $\pi^{2c}(t_1^2 - t_3^2)$, it represents all elements with valuation at least $2c$ and thus π^{2c+1} and $r\pi^{2c+1}$. If for $d \in R$ the equation $x_1^2 - r\pi^{2b} x_2^2 - \pi^{2c} x_3^2 = d$ is solvable, so is $x_1^2 - r\pi^{2b} x_2^2 - \pi^{2b} x_3^2 = d$, so we may assume $c = b$.

• If $x_1^2 - r\pi^{2b} x_2^2 - \pi^{2c} x_3^2 = r\pi^{2b-2}$ is solvable, then so is $x_1^2 - r\pi^{2b} x_2^2 - \pi^{2b} x_3^2 = r\pi^{2b-2}$. Then $\pi^{b-1} \mid x$, so

$$X_1^2 - r\pi^2 x_2^2 - \pi^2 x_3^2 = r,$$

and going modulo π gives a contradiction.

• If $x_1^2 - r\pi^{2b} x_2^2 - \pi^{2c} x_3^2 = u\pi^{2c-1}$ is solvable, then so is

$$x_1^2 - rx_2^2 - \pi^{2c} x_3^2 = u\pi^{2c-1}.$$

Since $x_1^2 - rx_2^2$ is anisotropic modulo π, we get $x_1 = \pi X_1$, $y_1 = \pi Y_1$ and

$$\pi X_1^2 - r\pi X_2^2 - \pi^{2c-1} x_3^2 = u\pi^{2c-2}.$$

If $c = 1$, the left hand side is divisible by π and the right hand side is not, contradiction. Otherwise:

$$X_1^2 - rX_2^2 - \pi^{2c-2}x_3^2 = u\pi^{2c-3},$$

and by induction on c we get a contradiction.

Case 3.2.2: $f = t_1^2 - r\pi^{2b}t_2^2 - r\pi^{2c}t_3^2$ with $b \le c$.
The representation table is $(0, 2b, 2c+1, 2c+1)$; the computations are similar to those of Case 3.2.1.

Case 3.2.3: $f = t_1^2 - r\pi^{2b}t_2^2 + \pi^{2c+1}t_3^2$ with $b \le c$.
We claim that the representation table is $(0, 2b, 2c+1, \infty)$. Since f is anisotropic, it does not represent $-\operatorname{disc} f \pmod{K^{\times 2}} = r\pi \pmod{K^{\times 2}}$. As above it represents $r\pi^{2b}$ and clearly it represents π^{2c+1}.
• If $x^2 - r\pi^{2b}y^2 + \pi^{2c+1}z^2 = r\pi^{2b-2}$ is solvable, then so is

$$x^2 - r\pi^{2b}y^2 + \pi z^2 = r\pi^{2b-2}.$$

If $b = 1$ then going modulo π gives a contradiction. Otherwise we may take $x = \pi X$, getting

$$\pi X^2 - r\pi^{2b-1}y^2 + z^2 = r\pi^{2b-3}$$

and then $z = \pi Z$, getting

$$X^2 - r\pi^{2b-2}y^2 + \pi Z^2 = r\pi^{2b-4},$$

and by induction on b we get a contradiction.
• If $x^2 - r\pi^{2b}y^2 + \pi^{2c+1}z^2 = \pi^{2c-1}$ is solvable, then so is

$$x^2 - ry^2 + \pi^{2c+1}z^2 = \pi^{2c-1}.$$

Going modulo π shows we can take $x = \pi X$, $y = \pi Y$, getting

$$\pi X^2 - r\pi Y^2 + \pi^{2c}z^2 = \pi^{2c-2}.$$

If $c = 1$ then going modulo π gives a contradiction; else we get

$$X^2 - rY^2 + \pi^{2c-1}z^2 = \pi^{2c-3},$$

and by induction on c we get a contradiction.

Case 3.2.4: $f = t_1^2 - r\pi^{2b}t_2^2 + r\pi^{2c+1}t_3^2$ with $b \le c$.
The representation table is $(0, 2b, \infty, 2c+1)$; the computations are similar to those of Case 3.2.3.

Case 3.3.1: $f = t_1^2 + \pi^{2b+1}t_2^2 - \pi^{2c}t_3^2$ with $b < c$.
We claim that the representation table is $(0, 2c, 2b+1, 2c+1)$. By Case 2.1, f represents $r\pi^{2c}$; clearly it represents π^{2b+1}. Also f represents $\pi^{2c}(t_1^2 - t_3^2)$ so represents $r\pi^{2c+1}$.
• If $f = x_1^2 + \pi^{2b+1}y^2 - \pi^{2c}z^2 = r\pi^{2c-2}$ is solvable, then so is

$$x^2 + \pi y^2 - \pi^{2c}z^2 = r\pi^{2c-2}.$$

If $c = 1$, then going modulo π yields a contradiction. Else we may take $x = \pi X$, getting

$$\pi X^2 + y^2 - \pi^{2c-1}z^2 = r\pi^{2c-3}$$

and then $y = \pi Y$, getting

$$X^2 + \pi Y^2 - \pi^{2c-2}z^2 = r\pi^{2c-4},$$

and by induction on c we get a contradiction.
• If $f = x^2 + \pi^{2b+1}y^2 - \pi^{2c}z^2 = \pi^{2b-1}$ is solvable, then we may take $x = \pi^b X$, getting

$$\pi X^2 + \pi^2 y^2 - \pi^{2c-2b+1}z^2 = 1,$$

and going modulo π gives a contradiction.
• If $f = x_1^2 + \pi^{2b+1}y^2 - \pi^{2c}z^2 = r\pi^{2c-1}$ is solvable, then so is

$$f = x^2 + \pi y^2 - \pi^{2c}z^2 = r\pi^{2c-1}.$$

We may take $x = \pi X$, getting

$$\pi X^2 + y^2 - \pi^{2c-1} z^2 = r\pi^{2c-2}.$$

If $c = 1$, then going modulo π gives a contradiction. Else we can take $y = \pi Y$, getting

$$X^2 + \pi y^2 - \pi^{2c-2} z^2 = r\pi^{2c-3},$$

and by induction on c we get a contradiction.

Case 3.3.2: $f = t_1^2 + \pi^{2b+1} t_2^2 - \pi^{2c+1} t_3^2$ with $b \le c$.
The representation table is $(0, 2c+2, 2b+1, 2c+1)$; the computations are similar to those of Case 3.3.1.

Case 3.3.3: $f = t_1^2 + \pi^{2b+1} t_2^2 - r\pi^{2c} t_3^2$ with $b < c$.
We claim that the representation table is $(0, 2c, 2b+1, \infty)$.

Since f is anisotropic, it does not represent

$$-\operatorname{disc} f \pmod{K^{\times 2}} = r\pi \pmod{K^{\times 2}}.$$

Clearly f represents π^{2b+1}; it also represents $\pi^{2c}(t_1^2 - rt_3^2)$ hence $\pi^{2c} r$.
• If $f = x^2 + \pi^{2b+1} y^2 - r\pi^{2c} z^2 = r\pi^{2c-2}$ is solvable, then so is

$$x^2 + \pi y^2 - r\pi^{2c} z^2 = r\pi^{2c-2}.$$

If $c = 1$, then going modulo π gives a contradiction. Else we may take $x = \pi X$, getting

$$\pi X^2 + y^2 - r\pi^{2c-1} z^2 = r\pi^{2c-3},$$

and then $y = \pi Y$, getting

$$X^2 + \pi Y^2 - r\pi^{2c-2} z^2 - r\pi^{2c-4},$$

and by induction on c we get a contradiction.
• If $f = x^2 + \pi^{2b+1} y^2 - r\pi^{2c} z^2 = \pi^{2b-1}$ is solvable, then so is

$$x^2 + \pi^{2b+1} y^2 - r\pi^{2b} z^2 = \pi^{2b-1}.$$

We may take $x = \pi X$, getting

$$\pi X^2 + \pi^{2b} y^2 - r\pi^{2b-1} z^2 = \pi^{2b-2}.$$

If $b = 1$, then going modulo π gives a contradiction. Else we get

$$X^2 + \pi^{2b-1} y^2 - r\pi^{2b-2} z^2 = \pi^{2b-3},$$

and by induction on b we get a contradiction.

Case 3.3.4: $f = t_1^2 + \pi^{2b+1} t_2^2 - r\pi^{2c+1} t_3^2$ with $b \le c$.
The representation table is $(0, \infty, 2b+1, 2c+1)$; the computations are similar to those of Case 3.3.3.

Case 3.4.1: $f = t_1^2 + r\pi^{2b+1} t_2^2 - \pi^{2c} t_3^2$ with $b < c$.
The representation table is $(0, 2c, 2c+1, 2b+1)$; the computations are similar to those of Case 3.3.1.

Case 3.4.2: $f = t_1^2 + r\pi^{2b+1} t_2^2 - r\pi^{2c+1} z^2$ with $b \le c$.
The representation table is $(0, 2c+2, 2c+1, 2b+1)$; the computations are similar to those of Case 3.3.1.

Case 3.4.3: $f = t_1^2 + r\pi^{2b+1} t_2^2 - r\pi^{2c} t_3^2$ with $b < c$.
The representation table is $(0, 2c, \infty, 2b+1)$; the computations are similar to those of Case 3.3.3.

Case 3.4.4: $f = t_1^2 + r\pi^{2b+1} t_2^2 - \pi^{2c+1} t_3^2$ with $b \le c$.
The representation table is $(0, \infty, 2c+1, 2b+1)$; the computations are similar to those of Case 3.3.3.

From this we find that for a ternary form $x^2+by^2+cz^2$ the possible representation tables are precisely the following: for any $b,c\in\mathbb{N}$,

$$(0,\infty,2b+1,2c+1),(0,2b,\infty,2c+1),(0,2b,2c+1,\infty),$$
$$(0,2b,2b+1,2b+1),(0,2b,2c+1,2c+1),(0,2b,2c+1,2b+1),$$

and also:

$$\forall b,c\in\mathbb{N} \text{ with } b\le c,\ (0,2b,2c+1,2c+1),\ (0,2c+2,2c+1,2b+1)$$
$$\forall b,c\in\mathbb{N} \text{ with } c< b,\ (0,2b,2c+1,2b+1),\ (0,2b,2b+1,2c+1)$$
$$\forall b,c\in\mathbb{N} \text{ with } c\le b,\ (0,2b+2,2c+1,2b+1),\ (0,2b+2,2b+1,2c+1).$$

2.2.3. *Quaternary forms.* Let $f_{/R}$ be a quaternary form. Then f is K-universal, so f R-represents elements of every K-adic square class, and thus the representation table is of the form $(2a,2b,2c{+}1,2d{+}1)$ for $a,b,c,d\in\mathbb{N}$. We claim that all of these representation tables actually occur. As in Remark 2.4, via scaling it is enough to show that all representation tables $(0,2b,2c+1,2d+1)$ occur. This is accomplished by the following calculations.

Case 4.1: $f: t_1^2-r\pi^{2b}t_2^2+r\pi^{2d+1}t_3^2-\pi^{2c+1}t_4^2$ with $d\le c$.
We claim that the representation table is $(0,2b,2c+1,2d+1)$.

By Case 2.2, f represents $r\pi^{2b}$. Clearly f represents $r\pi^{2d+1}$. Moreover f represents $\pi^{2c+1}(rt_3^2-t_4^2)$ hence it represents π^{2c+1}.

- If $f=x^2-r\pi^{2b}y^2+r\pi^{2d+1}z^2-\pi^{2c+1}w^2=r\pi^{2b-2}$ is solvable, then so is

$$x^2-r\pi^{2b}y^2+r\pi z^2-\pi w^2=r\pi^{2b-2}.$$

If $b=1$, then going modulo π gives a contradiction. Else we may take $x=\pi X$, getting

$$\pi X^2-r\pi^{2b-1}y^2+rz^2-w^2=r\pi^{2b-3}.$$

Going modulo π shows that we may take $z=\pi Z$ and $w=\pi W$, getting

$$X^2-r\pi^{2b-2}y^2+r\pi Z^2-\pi W^2=r\pi^{2b-4},$$

and by induction on b we get a contradiction.

- If $f=x^2-r\pi^{2b}y^2+r\pi^{2d+1}z^2-\pi^{2c+1}w^2=\pi^{2c-1}$ is solvable, then so is

$$x^2-ry^2+r\pi z^2-\pi^{2c+1}w^2=\pi^{2c-1}.$$

Going modulo π shows that we may take $x=\pi X$ and $y=\pi Y$, getting

$$\pi X^2-r\pi Y^2+rz^2-\pi^{2c}w^2=\pi^{2c-2}.$$

If $c=1$, then going modulo π gives a contradiction. Else we may take $z=\pi Z$, getting

$$X^2-rY^2+r\pi Z^2-\pi^{2c-1}w^2=\pi^{2c-3},$$

and by induction on c we get a contradiction.

- If $f=x^2-r\pi^{2b}y^2+r\pi^{2d+1}z^2-\pi^{2c+1}w^2=r\pi^{2d-1}$ is solvable, then so is

$$x^2-ry^2+r\pi^{2d+1}z^2-\pi^{2d+1}w^2=r\pi^{2d-1}.$$

Going modulo π shows that we may take $x=\pi X$ and $y=\pi Y$, getting

$$\pi X^2-r\pi Y^2+r\pi^{2d}z^2-\pi^{2d}w^2=r\pi^{2d-2}.$$

If $d=1$, then going modulo π gives a contradiction. Else we get

$$X^2-rY^2+r\pi^{2d-1}z^2-\pi^{2d-1}w^2=r\pi^{2d-3}.$$

Going modulo π shows that we may take $X=\pi\mathcal{X}$ and $Y=\pi\mathcal{Y}$, getting

$$\pi\mathcal{X}^2-r\pi\mathcal{Y}^2+r\pi^{2d-2}z^2-\pi^{2d-2}w^2=r\pi^{2d-4},$$

and by induction on d we get a contradiction.

Case 4.2: $f : t_1^2 - r\pi^{2b}t_2^2 + \pi^{2c+1}t_3^2 - r\pi^{2d+1}t_4^2$ with $c \leq d$.
The representation table is $(0, 2b, 2c+1, 2d+1)$; the computations are similar to those of Case 4.1.

2.2.4. $n \geq 5$. Suppose that $f_{/R}$ is an n-ary quadratic form with $n \geq 5$.

Case 5.1: Let $b, c, d \in \mathbb{N}$ with $d \leq c$, and put $A := \max(2b-1, 2c, 2d)$. Let $f : t_1^2 - r\pi^{2b}t_2^2 + r\pi^{2d+1}t_3^2 - \pi^{2c+1}t_4^2 + \pi^A(t_5^2 + \ldots + t_n^2)$. Revisiting the argument of Case 4.1, we see that the representation table remains $(0, 2b, 2c+1, 2d+1)$.

Case 5.2: Let $b, c, d \in \mathbb{N}$ with $c \leq d$, and put $A := \max(2b-1, 2c, 2d)$. Let $f : t_1^2 - r\pi^{2b}t_2^2 + \pi^{2c+1}t_3^2 - r\pi^{2d+1}t_4^2 + \pi^A(t_5^2 + \ldots + t_n^2)$. Revisiting the argument of Case 4.1, we see that the representation table remains $(0, 2b, 2c+1, 2d+1)$.

2.2.5. *A consequence.*

Theorem 2.5. *Let R be a complete DVR with residue field of finite odd cardinality q. Let $f_{/R}$ be an n-ary anisotropic ADC form.*

a) *If $n = 2$, then $\delta_v(f) \in \{\frac{q}{q+1}, \frac{1}{2}, \frac{1}{q+1}\}$, and all of these occur.*
b) *If $n = 3$, then $\delta_v(f) \in \{\frac{q+2}{2q+2}, \frac{2q+1}{2q+2}\}$, and both of these occur.*
c) *If $n \geq 4$, then $\delta_v(f) = 1$.*

Proof. We have $v_1, v_r \in \{0, \infty\}$ and $v_\pi, v_{r\pi} \in \{1, \infty\}$. So if f represents a out of the square classes $\{1, r\}$ and b out of the square classes $\{\pi, r\pi\}$ then by (4) we have

$$\delta_v(f) = \frac{aq + b}{2q + 2}.$$

a) If $n = 2$, then we know that $a + b = 2$, so $(a, b) \in \{(2, 0), (1, 1), (0, 2)\}$, which leads, respectively, to $\delta_v(f) = \frac{q}{q+1}$, $\delta_v(f) = \frac{1}{2}$ and $\delta_v(f) = \frac{1}{q+1}$. Moreover, there are $6 = \binom{4}{2}$ inequivalent anisotropic binary forms over K and every pair of square classes is represented by exactly one of these forms. So all of these densities occur.

b) If $n = 3$, then f represents all square classes except $-\operatorname{disc} f$. Thus we have $(a, b) \in \{(2, 1), (1, 2)\}$, which leads, respectively, to $\delta_v(f) \in \{\frac{2q+1}{2q+2}, \frac{q+2}{2q+2}\}$. Moreover, $-\operatorname{disc} f$ can be any square class, just by scaling any one anisotropic ternary form. So all of these densities occur.

c) If $n \geq 4$ then f is K-universal and ADC hence R-universal. So $\delta_v(f) = 1$. □

2.3. **ADC forms over $\mathbb{Z}_2$.** Let $f_{/\mathbb{Z}_2}$ be an n-ary ADC form. The ADC condition implies that for all $s \in \{1, 3, 5, 7\}$ we have $v_s \in \{0, \infty\}$, while for all $s \in \{2, 6, 10, 14\}$ we have $v_s \in \{1, \infty\}$. Thus if f represents a of the square classes $\{1, 3, 5, 7\}$ and b of the square classes $\{2, 6, 10, 14\}$, by (4) we have

$$\delta_v(f) = \frac{2a + b}{12}.$$

If $n \geq 4$ then f is $\mathbb{Q}_2$-universal and ADC, so it is $\mathbb{Z}_2$-universal and $\delta_2(f) = 1$. Moreover if f is isotropic and ADC, then it is $\mathbb{Z}_2$-universal and $\delta_2(f) = 1$. So the nontrivial cases (among ADC forms) are when $n \in \{2, 3\}$ and f is anisotropic.

Recall also that in the complete local case maximal lattices are ADC, so every $\mathbb{Q}_2$-isomorphism class of quadratic forms yields at least one $\mathbb{Z}_2$-isomorphism class of ADC forms.

2.3.1. *Binary forms.* Let $f_{/\mathbb{Z}_2}$ be an anisotropic ADC binary form. By [L], $f_{/\mathbb{Q}_2}$ represents precisely 4 of the 8 elements of $\mathcal{S}$. Thus $a + b = 4$. One sees – e.g. by a brute force search of the 36 binary forms $ax^2 + by^2$ obtained by letting a and b run through unordered pairs of square classes in $\mathbb{Q}_2$ – that the pairs $(1, 3)$ and $(3, 1)$ do not occur. As for the others:

$2x^2 + 2xy + 2y^2$ is ADC and $\mathbb{Q}_2$-represents $2, 6, 10, 14$, so $(a, b) = (0, 4)$, and $\delta_2(f) = \frac{1}{3}$.

$x^2 + y^2$ $\mathbb{Q}_2$-represents $1, 2, 5, 10$, hence $(a, b) = (2, 2)$, and $\delta_2(f) = \frac{1}{2}$.
$x^2 + xy + y^2$ is ADC and $\mathbb{Q}_2$-represents $1, 3, 5, 7$, hence $(a, b) = (4, 0)$, and $\delta_2(f) = \frac{2}{3}$.

2.3.2. *Ternary forms.* Let $f_{/\mathbb{Z}_2}$ be an anisotropic ADC ternary form. Then f fails to represent the $\mathbb{Q}_2$-square class of $-\operatorname{disc} f$ and represents all other square classes. By scaling we see that $-\operatorname{disc} f$ can be any square class. So the possibilities are

$$\delta_2(f) = \frac{5}{6},\ \delta_2(f) = \frac{11}{12}.$$

2.4. **Isotropic binary forms over $\mathbb{Z}_2$.** Let

$$f(t_1, t_2) = At_1^2 + Bt_1t_2 + Ct_2^2$$

be a primitive isotropic binary form over $\mathbb{Z}_2$ with discriminant $\Delta = B^2 - 4AC$. Then Δ is a square in $\mathbb{Z}_2$, so $a := \frac{1}{2}v_2(\Delta) \in \mathbb{N}$. We will compute $\delta_2(f)$ in terms of a. Since every binary form over $\mathbb{Z}_2$ is either diagonalizable or $\mathbb{Z}_2$-equivalent to $2^a(t_1^2 + t_1t_2 + t_2^2)$ or $2^a t_1 t_2$ for some $a \in \mathbb{N}$ [Ca, Lemma 8.4.1], and since f is primitive with square discriminant, f is isomorphic to either t_1t_2 or $u(t_1^2 - 2^{2a-2}t_2^2)$ for $u \in \mathbb{Z}_2^\times$. The former case occurs if and only if $a = 0$, and in this case clearly $\delta_2(f) = 1$. In the latter case we have $\delta_2(f) = \delta_2(t_1^2 - 2^{2a-2}t_2^2)$. If $a = 1$ then $\delta_2(f) = \delta_2(t_1^2 - t_2^2)$. It is easy to see that f represents precisely the elements of $\mathbb{Z}_2$ with valuation different from 1, so $\delta_2(f) = \frac{3}{4}$. Now we suppose that $a \geq 2$. Ordering the $\mathbb{Q}_2$ square classes as $1, 3, 5, 7, 2, 6, 10, 14$, we claim that the representation table of f is

$$(0, 2a - 2, 2a - 4, 2a - 2, 2a + 1, 2a + 1, 2a + 1, 2a + 1).$$

First suppose that $a = 2$. Then one easily sees that $t_1^2 - 4t_2^2$ has representation table $(0, 2, 0, 2, 5, 5, 5, 5)$. Now suppose $a \geq 3$. Clearly $t_1^2 - 2^{2a-2}t_2^2$ represents 1; reducing modulo 8 shows that it does not represent any of $2, 3, 5, 6, 7, 10, 14$. Now, for $w \in \mathbb{Z}_p$, suppose that there are $x, y \in \mathbb{Z}_2$ such that $x^2 - 2^{2a-2}y^2 = 4w$. Then we may take $x = 2X$ and get

$$X^2 - 2^{2a-4}y^2 = w,$$

and conversely if $t_1^2 - 2^{2a-4}t_2^2$ represents w then $t_1^2 - 2^{2a-2}t_2^2$ represents $4w$. So if the representation table for $t_1^2 - 2^{2a-4}t_2^2$ is

$$(0, 2a - 4, 2a - 6, 2a - 4, 2a - 1, 2a - 1, 2a - 1, 2a - 1)$$

then the representation table for $t_1^2 - 2^{2a-2}t_2^2$ is

$$(0, 2a - 2, 2a - 4, 2a - 2, 2a + 1, 2a + 1, 2a + 1, 2a + 1),$$

and the claim follows by induction on a.

Combining the above analysis with (4), we get the following result.

Proposition 2.6. *Let $f_{/\mathbb{Z}_2}$ be a primitive isotropic binary quadratic form of Discriminant Δ, and let $a = \frac{1}{2}v_2(\Delta)$. Then:*
a) If $a = 0$, then $\delta_2(f) = 1$.
b) If $a = 1$, then $\delta_2(f) = \frac{3}{4}$.
c) If $a \geq 2$, then $\delta_2(f) = \frac{2 + 2^{4-2a} + 2^{5-2a} + 2^{2-2a}}{12}$.

2.5. **A consequence.** The following is an immediate consequence of our results.

Theorem 2.7. *Let $f_{/\mathbb{Z}_p}$ be an n-ary ADC form (e.g. a maximal lattice). Suppose moreover that either $n = 3$ or $p > 2$. Then either $\delta_p(f) = 1$ or $\delta_p(f)$ is a rational number with negative 2-adic valuation.*

2.6. **Some examples.**

Example 2.8. Let $p > 2$ and let $f = x^2 + py^2 - pz^2$. The binary subform $py^2 - pz^2 = p(y^2 - z^2)$ is isotropic, hence so is f. Since $y^2 - z^2 \cong yz$ is universal, the form f represents every square class with valuation at least 1. Clearly it also represents $\mathbb{Z}_p^{\times 2}$, but it does not represent the unit non-residue r: for all $x, y, z \in \mathbb{Z}_p$, $x^2 + py^2 - pz^2 \pmod p \in \mathbb{F}_p^{\times 2} \cup \{0\}$. We conclude

$$\delta_p(f) = 1 - \frac{p-1}{2p} = \frac{p+1}{2p}.$$

Thus $v_2(\delta_p(f))$ is non-negative for all $p > 2$, is strictly positive if and only if $p \equiv 3 \pmod 4$ and is arbitrarily large on a set of primes of positive relative density.

Example 2.9. Let $n \geq 3$, $p > 2$ and put

$$f = t_1^2 + p^2 t_2^2 - p^2 t_3^2 - \ldots - p^2 t_n^2.$$

Then the subform $p^2t_2^2 - p^2t_3^2$ is isotropic and represents all x with $v(x) \geq 2$. Moreover f does represent 1 and does not represent any of r, p, rp. So

$$\delta_p(f) = 1 - \frac{p-1}{2p} - \frac{p-1}{2p^2} - \frac{p-1}{2p^2} = \frac{p^2 - p + 2}{2p^2}.$$

Example 2.10. Let $n \geq 4$ and let $p > 2$. Choose $a_1, a_2 \in \mathbb{Z}_p^\times$ such that $-a_1a_2 \equiv r \pmod{\mathbb{Z}_p^{\times 2}}$. Let $f_0 = a_1t_1^2 + a_2t_2^2 + pt_3^2$ and

$$f(t_1, \ldots, t_n) := f_0(t_1, t_2, t_3) + p^2 t_4^2 + \ldots + p^2 t_n^2.$$

The ternary subform f_0 is maximal and anisotropic, so represents precisely those elements of $\mathbb{Z}_p$ that do not lie in the $\mathbb{Q}_p$-adic square class of rp. If $x_1, \ldots, x_n \in \mathbb{Z}_p$ are such that

$$f(x_1, \ldots, x_n) = a_1x_1^2 + a_2x_2^2 + px_3^2 + p^2x_4^2 + \ldots + p^2x_n^2 = rp,$$

then $a_1x_1^2 + a_2x_2^2 \equiv 0 \pmod p$ and thus there are $X_1, X_2 \in \mathbb{Z}_p$ with $x_1 = pX_1$, $x_2 = pX_2$. Making this substitution and simplifying, we get

$$pa_1X_1^2 + pa_2X_2^2 + x_3^2 + px_4^2 + \ldots + px_n^2 = r.$$

Reducing mod p gives $x_3^2 \equiv r \pmod p$: contradiction. So f does not represent rp.

Since f_0 represents $p^2(rp-1)$ and $p^2t_4^2 + \ldots + p^2t_n^2$ represents p^2, f represents $p^2(rp-1) + p^2 = rp^3$, hence also rp^5, rp^7 and so forth. Thus f represents everything but the $\mathbb{Z}_p$-square class of rp, so

$$\delta_p(f) = 1 - \frac{p-1}{2p^2}.$$

3. Proof of the Product Formula

Suppose first that $n = 1$. Then for all but finitely many primes p, we have

$$\delta_p(f) = \frac{p-1}{2p}\left(1 + 1/p^2 + 1/p^4 + \ldots\right) = \frac{p}{2(p+1)} < 1/2,$$

so $\prod_p \delta_p(f) = 0$. Clearly, $\delta(f) = 0$ in this case as well, so the formula (2) holds. If $n = 2$ and $f_{/\mathbb{Z}}$ is anisotropic, again both sides of (2) vanish, as shown in §4.2.

Thus, we may assume that either $n \geq 3$, or that $n = 2$ and $f_{/\mathbb{Z}}$ is isotropic. In either case, we may fix a finite set of primes S such that $f_{/\mathbb{Z}_p}$ is universal for all $p \notin S$. For if $n \geq 3$, we have seen already that $\delta_p = 1$ for all odd primes not dividing the discriminant. When $n = 2$ and f is isotropic, then for all odd primes not dividing the discriminant, $f_{/\mathbb{Z}_p}$ is isomorphic to the hyperbolic plane and hence universal. Since $\delta_f = \delta_{\text{loc}}(f)$ (the Density Hasse Principle), to complete the proof of (2) it will suffice to show that

$$\delta_{\text{loc}}(f) = \prod_{p \in S} \delta_p(f). \tag{5}$$

Let $p \in S$. Asking that the nonzero integer n be represented by $f_{/\mathbb{Z}_p}$ is equivalent to requiring that n lie in one of the $\mathbb{Z}_p$-square classes represented by f. For $K \in \mathbb{Z}^+$, let $U_{p,K}$ be the union of the represented $\mathbb{Z}_p$-square classes whose p-adic valuation is smaller than K. Then $U_{p,K} \cap \mathbb{Z}$ is a union of residue classes modulo p^{K+2}, and the density of $U_{p,K} \cap \mathbb{Z}$ coincides with the Haar measure $\delta_{p,K}(f)$ (say) of $U_{p,K}$.

It now follows from the Chinese Remainder Theorem that the set of n that are locally represented by f and not divisible by p^K for any $p \in S$ has density

$$\delta_K(f) := \prod_{p \in S} \delta_{p,K}(f).$$

In particular, the set of locally represented integers has lower density bounded below by $\delta_K(f)$, for any K. Now letting $K \to \infty$, and noting that

$$\delta_{p,K}(f) \nearrow \delta_p(f) \qquad \text{for each fixed } p \in S,$$

we deduce that the set of locally represented integers has lower density at least $\prod_{p \in S} \delta_p(f)$. In the other direction, the upper density of locally represented integers is at most

$$\delta_K(f) + \sum_{p \in S} \frac{1}{p^K} \leq \prod_{p \in S} \delta_p(f) + \sum_{p \in S} \frac{1}{p^K}.$$

Letting $K \to \infty$ bounds this upper density from above by $\prod_{p \in S} \delta_p(f)$. This completes the proof of (5) (and also (2)).

4. Proof of Near Regularity

4.1. $n = 1$**.** For $a \in \mathbb{Z}^\bullet$, let $f = at^2$. It follows from the Global Square Theorem [L, Thm. VI.3.7] that $D_f = D_{f,\mathrm{loc}}$.

4.2. $n = 2$**, anisotropic case.** Let $f = ax^2 + bxy + cy^2$ be an integral binary quadratic form, with discriminant $ac - \frac{b^2}{4}$ and Discriminant $\Delta = b^2 - 4ac$. Then f is isotropic if and only if $\Delta = B^2$ for some $B \in \mathbb{N}$.

First we suppose that f is anisotropic. Bernays [Be12] showed that there is a $\kappa_\Delta > 0$ such that

$$\#D_f(X) \sim \frac{\kappa_\Delta X}{\log^{1/2} X}.$$

That $\#\mathcal{E}_f(X) = o(D_f(X))$ is also essentially due to Bernays. A more explicit and modern treatment is given in work of Odoni [Od77, §5].

It is much easier to show that $\delta(f) = \delta_{\mathrm{loc}}(f) = 0$. The set $\mathcal{S}$ of prime numbers $p > 2$ such that $\left(\frac{-\operatorname{disc}(f)}{p}\right) = -1$ has density $\frac{1}{2}$ within the primes. For each such prime, $f_{/\mathbb{Z}_p}$ is anisotropic and represents precisely the two unit square classes, so $\delta_p(f) = 1 - \frac{1}{p+1}$. Therefore

$$0 \leq \delta \leq \delta_{\mathrm{loc}}(f) \leq \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p+1}\right) = 0.$$

4.3. $n = 2$**, isotropic case.** Suppose f is isotropic, so $\Delta = B^2$ for some $B \in \mathbb{Z}^+$. We may assume without loss of generality that f is primitive. The form f is maximal if and only if $\Delta = 1$, in which case f is a maximal lattice in the hyperbolic plane, and by [Sh, Lemma 29.8] we have $f \cong_{\mathbb{Z}} xy$ and f is universal. Thus we may assume that $\Delta > 1$. Gauss showed [Ga, Art. 206] that f is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a form

$$Ax^2 + Bxy, \ 1 \leq A < B, \ \gcd(A, B) = 1. \tag{6}$$

Lemma 4.1. *Let a and k be coprime positive integers. Then almost all positive integers have a prime factor $p \equiv a \pmod{k}$ – i.e., the exceptional set has density zero in $\mathbb{Z}^+$.*

Proof. For each fixed $B \geq 1$, the upper density of the exceptional set is bounded above by the density of the set of $n \in \mathbb{Z}^+$ that are divisible by no prime $p \equiv a \pmod{k}$ with $p \leq B$, i.e., by

$$\prod_{p \leq B,\ p \equiv a \pmod{k}} \left(1 - \frac{1}{p}\right).$$

This quantity is at most

$$\exp\left(-\sum_{p \leq B,\ p \equiv a \pmod{k}} \frac{1}{p}\right).$$

The usual proof of Dirichlet's theorem on primes in arithmetic progressions gives

$$\sum_{p \equiv a \pmod{k}} \frac{1}{p} = \infty.$$

Taking $B \to \infty$, the result follows. □

Lemma 4.2. *As K appoaches infinity, the density of the set of integers that are* not *Kth powerfree approaches* 0.

Proof. We show the statement for the positive integers, which is sufficient. If $n \in \mathbb{Z}^+$ is not Kth power free then there is an integer $m \geq 2$ such that $m^K \mid n$. It follows that for each $x > 0$, the number of such $n \leq x$ that are not Kth power free is at most $\sum_{m=2}^{\infty} \frac{x}{m^K}$, and thus the upper density of the set of non-Kth-powerfree positive integers is at most $\sum_{m=2}^{\infty} \frac{1}{m^K}$. This quantity tends to zero as K approaches infinity, e.g. by the Integral Test. □

Let $f = Ax^2 + Bxy$ with $1 \leq A < B$ and $\gcd(x, y) = 1$. We will prove that almost every integer locally represented by f is globally represented. Taking $x = 1$ we see that D_f contains every element congruent to A modulo B, so $\#D_f(X) \gg X$. So an equivalent statement of the result is that $\#\mathcal{E}_f(X) = o(X)$. Fix $K \in \mathbb{Z}^+$, and let $\mathcal{E}_{f,K}(X)$ be the subset of $\mathcal{E}_f(X)$ consisting of integers that are Kth power free. By Lemma 4.2, it is enough to show that for all $K \geq 2$ we have $\#\mathcal{E}_{f,K}(X) = o_K(X)$. Put

$$M \coloneqq \prod_{p \mid B} p^{K + v_p(B)}.$$

Let $n \in \mathcal{E}_{f,K}(X)$. It follows from Lemma 4.1 that for a density 1 subset of $n \in \mathbb{Z}$ we have that for all c coprime to M there is a prime $p \mid n$ such that $p \equiv c \pmod{M}$. Henceforth we restrict to n in this subset and such that f locally represents n. For every prime $p \mid n$ there are $x_{0,p}, y_{0,p} \in \mathbb{Z}_p$ such that

$$x_{0,p}(Ax_{0,p} + By_{0,p}) = n. \tag{7}$$

Step 1: We show that there is $x \in \mathbb{Z}$ such that

$$x \mid n \text{ and } \forall p \mid B,\ x \equiv x_{0,p} \pmod{p^{K+v_p(B)}}. \tag{8}$$

To see this, let $r \in \mathbb{Z}$ be such that $r \equiv x_{0,p} \pmod{p^{K+v_p(B)}}$ for all $p \mid B$. Let $p \mid B$. It follows from (7) that

$$v_p(x_{0,p}) \leq v_p(n) \leq K$$

and thus

$$v_p(r) = v_p(x_{0,p}).$$

It follows that there is an integer u coprime to M such that $u \prod_{p \mid B} p^{v_p(x_{0,p})} \equiv r \pmod{M}$. Because of our assumption on n, we may take u to be the class in $\mathbb{Z}/M\mathbb{Z}$ of a prime $q \mid n$, and thus

$$x \coloneqq q \prod_{p \mid B} p^{v_p(x_{0,p})}$$

satisfies (8).

Step 2: Let $y_{1,p} \in \mathbb{Z}$ be congruent to $y_{0,p}$ modulo $p^{K+v_p(B)}$. Reducing (7) modulo $p^{K+v_p(B)}$, we get

$$x(Ax + By_{1,p}) \equiv n \pmod{p^{K+v_p(B)}}.$$

Since $v_p(x) \leq v_p(n) \leq K$, we get

$$Ax \equiv Ax + By_{1,p} \equiv \frac{n}{x} \pmod{p^{v_p(B)}}.$$

Since this holds for all $p \mid B$ we have

$$Ax \equiv \frac{n}{x} \pmod{B}.$$

Thus we can write $\frac{n}{x} = Ax + By$ for some $y \in \mathbb{Z}$, so $n = x(Ax + By) = f(x, y)$.

Example 4.3. Let $f(x, y) = x^2 + 5xy$. If $\ell \equiv 4 \pmod 5$ is a prime number, then f $\mathbb{Z}_p$-represents ℓ for all primes p: for $p \neq 5$, there is a representation with $x = 1$; over $\mathbb{Z}_5$ there is a representation with $y = 0$. However, it is easy to check that the equation $x(x + 5y) = \ell$ has no solutions $(x, y) \in \mathbb{Z}^2$. Thus f is not almost regular, and Dirichlet's theorem on primes in arithmetic progressions gives $\mathcal{E}_f(X) \gg \frac{X}{\log X}$.

4.4. $n = 3$**, positive case.** Let $f_{/\mathbb{Z}}$ be a positive r-ary quadratic form. The nth coefficient of the Eisenstein projection of the theta series of f can be written as

$$a_E(n) = \prod_{p \leq \infty} \beta_p(n)$$

where for finite p,

$$\beta_p(n) = \lim_{k \to \infty} \frac{\#\{x \in (\mathbb{Z}/p^k\mathbb{Z})^r : f(x) \equiv n \pmod{p^k}\}}{p^{(r-1)k}},$$

and if we write $f(t) = t^T At$, then

$$\beta_\infty(n) = \frac{r\pi^{r/2} n^{(r-2)/2}}{2\Gamma\left(\frac{r}{2} + 1\right) \det(A)^{1/2}}.$$

For a prime p, we write the local Jordan splitting of f (as in [Ha04]) as

$$f(x) = \sum_j p^{v_j} f_j(x_j).$$

If $p > 2$, each f_j is one-dimensional, while f_j can be two-dimensional if $p = 2$. We let $r_{p^k,f}(m) = \#\{x \in (\mathbb{Z}/p^k\mathbb{Z})^r : f(x) \equiv m \pmod{p^k}\}$. A solution to this congruence is called *good type* if $p^{v_j} x_j \not\equiv 0 \pmod p$ for some j. It is called *zero type* if $x \equiv 0 \pmod p$, and *bad type* otherwise. In [Ha04], Hanke gives a recursive method to determine the number of good type, bad type, and zero type solutions to $f(x) \equiv m \pmod{p^k}$.

Proposition 4.4. *Let $f_{/\mathbb{Z}}$ be a positive ternary quadratic form. As $X \to \infty$ we have $\mathcal{E}_f(X) = O(\sqrt{X})$.*

Proof. Let $\theta_f(z) := \sum_{n=0}^{\infty} r_f(n) q^n, \quad q = e^{2\pi i z}$ be the theta series of f. We may decompose

$$\theta_f(z) = E(z) + H(z) + C(z) = \sum_{n=0}^{\infty} a_E(n) q^n + \sum_{n=0}^{\infty} a_H(n) q^n + \sum_{n=0}^{\infty} a_C(n) q^n.$$

Here $E(z)$ is a weight 3/2 Cohen-Eisenstein series, $H(z)$ is a weight 3/2 cusp form that is a linear combination of unary theta series (modular forms of the shape $\sum_{n \in \mathbb{Z}} \psi(n) n q^{n^2}$ where ψ is an odd Dirichlet character), and $C(z)$ is the projection of $\theta_f(z)$ onto the orthogonal complement in the space of weight 3/2 cusp forms of unary theta series. Let $N(f)$ be the level of f.

Cohen-Eisenstein series: It follows from a multivariate form of Hensel's lemma that $\beta_p(n) > 0$ if and only if there is a solution to $f(x) = n$ in $\mathbb{Z}_p$.

If $p \nmid N(f)$ and $p \nmid n$, Hanke gives the formula

$$\beta_p(n) = 1 + \frac{1}{p}\left(\frac{-\det(A)n}{p}\right).$$

(See Table 1 on the top of page 363 of [Ha04].)

Now we consider the case that $p \nmid N(f)$ but $p \mid n$. In this case, $f_{/\mathbb{F}_p}$ is isotropic (because if f is anisotropic, then $p^2 \mid N(f)$). As a consequence, $f(x,y,z) = 0$ defines a conic over $\mathbb{F}_p$ which has a point on it. Therefore $f(x,y,z) = 0$ is isomorphic to $\mathbb{P}^1$ over $\mathbb{F}_p$ and there are $p+1$ (projective) points on it. This yields $(p+1)(p-1) = p^2 - 1$ solutions to the equation $f(x,y,z) \equiv 0 \pmod{p}$ with $(x,y,z) \not\equiv (0,0,0) \pmod{p}$. These are "good type" solutions in the terminology of Hanke, and from the recursive formula for good type solutions (Lemma 3.2 of [Ha04]) it follows that $\beta_p(n) \geq \frac{p^2-1}{p^2}$.

If $p \mid N(f)$ but $f_{/\mathbb{Q}_p}$ is isotropic, then there are finitely many possibilities for $\beta_p(n)$ if $v_p(n) < v_p(N(f))$. If $v_p(n) \geq v_p(N(f))$, then n is *p-stable* in the terminology of Hanke (see Definition 3.6 and Remark 3.6.1), and it follows that the contribution to $\beta_p(p^{2v}n)$ of the good type and bad type solutions is constant for $v \geq 1$. Moreover, by Lemma 3.8, this contribution is nonzero, and it follows that there is an absolute lower bound on $\beta_p(n)$ over all n that are locally represented by f.

Finally, if $f_{/\mathbb{Q}_p}$ is anisotropic, then again there are finitely many possibilities for $\beta_p(n)$ if $v_p(n) \leq v_p(N(f)) + 1$. The net result is that there is an absolute constant C_E so that

$$a_E(n) \geq C_E \beta_\infty(n) \prod_{\substack{p \nmid N \\ p \mid n}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p \nmid N \\ p \nmid n}} \left(1 + \frac{1}{p}\left(\frac{-\det(A)n}{p}\right)\right),$$

provided n is locally represented by f and $v_p(n) \leq v_p(N(f)) + 1$ for all anisotropic primes p.

We have that $\beta_\infty(n) = \frac{2\pi\sqrt{n}}{\det(A)^{1/2}}$. If we let χ_n be the unique primitive Dirichlet character with $\chi_n(p) = \left(\frac{-\det(A)n}{p}\right)$ for p prime with $p \nmid \det(A)n$, then we obtain

$$a_E(n) \geq \frac{12 C_E \sqrt{n}}{\pi} L(1, \chi_n) \prod_{p \mid nN} \left(1 + \frac{\chi_n(p)}{p}\right)^{-1}.$$

Now, we use the ineffective lower bound $L(1,\chi_n) \gg n^{-\epsilon}$. Also note that if k is squarefree, then $\prod_{p \mid k} \left(1 - \frac{1}{p}\right) = \frac{\phi(k)}{k} \gg k^{-\epsilon}$ for all $\epsilon > 0$. It follows from this that

$$a_E(n) \gg n^{1/2-\epsilon}$$

provided n is locally represented and $v_p(n) \leq v_p(N(f)) + 1$ for all anisotropic primes p.

Unary theta series: Because $H(z)$ is a linear combination of unary theta series, there are finitely many squarefree integers $b_1, b_2, \ldots, b_k$ so that $a_H(n) = 0$ unless n/b_i is the square of an integer.

Cusp form part: Theorem 1 of [Bl04] gives that for all $\epsilon > 0$

$$|a_C(n)| \ll_{f,\epsilon} n^{13/28+\epsilon}.$$

Since

$$r_f(n) = a_E(n) + a_H(n) + a_C(n),$$

the bounds above imply there are only finitely many n for which

(i) $a_H(n) = 0$,
(ii) n is locally represented, and
(iii) $v_p(n) \leq v_p(N(f)) + 1$ for all anisotropic primes p

which are not represented by f.

Let $c_1, c_2, \ldots, c_\ell$ be the list of all positive integers for which $a_H(n) = 0$, that are not represented by f, which are locally represented, and for which $v_p(c_i) \leq v_p(N(f)) + 1$ for all anisotropic primes p.

If n is a positive integer that is locally represented by f but not represented, one possibility is that $a_H(n) \neq 0$. In this case n/b_i is a square for some i and there are $O(\sqrt{x})$ many such $n \leq x$.

If n is a positive integer that is locally represented by f but not represented and $a_H(n) = 0$, let

$$S := \{p : p|n \text{ is anisotropic and } v_p(n) > v_p(N(f)) + 1\} = \{p_1, p_2, \ldots, p_s\}.$$

Corollary 3.8.2 of [Ha04] implies that if $n' = n / \prod_{i=1}^{s} p_i^{2v_i}$ is such that $v_p(n') = v_p(N(f))$ or $v_p(n') = v_p(N(f)) + 1$ for all $p \in S$, then $r_f(n) = 0 \implies r_f(n') = 0$. This n' must be one of the c_i mentioned above, and so $n/c_i = \prod_{i=1}^{s} p_i^{2v_i}$ is a perfect square. The number of such $n \leq x$ is $O((\log x)^t)$, where t is the number of anisotropic primes for f. $\square$

Example 4.5. The form $f(x, y, z) = 3x^2 + 4y^2 + 9z^2$ locally represents integers n with $v_2(n) \neq 1$ that are not in the $\mathbb{Q}_3$ square class of -3. We will show that the locally represented integers that are not represented are those of the form k^2 with all prime factors of $k \equiv 1 \pmod 3$. This fact was observed by Jones and Pall [JP39, Table II] and a proof of this is given by [SP80, Lemma 5]. We illustrate the techniques in the proof of Proposition 4.4 by giving a self-contained proof.

The decomposition of $\theta_f(z) = E(z) + H(z) + C(z)$ has $C(z) = 0$ and

$$H(z) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} n\chi_3(n) q^{n^2} - 2 \sum_{n=-\infty}^{\infty} n\chi_3(n) q^{4n^2}.$$

Here $\chi_3(n)$ is the non-principal Dirichlet character modulo 3. It follows that if n is not a square and n is locally represented, then $a_E(n) > 0$ and $a_H(n) = 0$ and so n is represented by f. If $n = k^2$ with k even, then $f(0, k/2, 0) = n$, and if $n = k^2$ with k a multiple of 3, then $f(0, 0, k/3) = n$.

Suppose that n is a square which is coprime to 6. Then the nth coefficient of $H(z)$ is $-n^{1/2}$. A straightforward computation with Yang's formulas for local densities [Ya98] shows that if n is a square coprime to 6, then

$$\beta_p(n) = \begin{cases} \frac{\pi\sqrt{3}n^{1/2}}{9} & \text{if } p = \infty, \\ 1 + \frac{1}{p} + (\chi_3(p) - 1) \cdot \frac{1}{p^{v_p(n)/2+1}} & \text{if } 3 < p < \infty, \\ 2 & \text{if } p = 3, \\ 1 & \text{if } p = 2. \end{cases}$$

This implies that

$$\begin{aligned} a_E(n) = \prod_{p \leq \infty} \beta_p(n) &= \frac{2\pi\sqrt{3}n^{1/2}}{9} \prod_{p>3} \left(1 + \frac{1}{p} + \frac{\chi_3(p) - 1}{p^{v_p(n)/2+1}}\right) \\ &= \frac{2\pi\sqrt{3}n^{1/2}}{9} \prod_{p \mid n} \left(1 + \frac{1}{p} + \frac{\chi_3(p) - 1}{p^{v_p(n)/2+1}}\right) \prod_{\substack{p>3 \\ p \nmid n}} \left(1 + \frac{\chi_3(p)}{p}\right) \\ &= \frac{\pi\sqrt{3}n^{1/2}}{3} \prod_p \frac{p^2 - 1}{p^2} \prod_{p \mid n} \frac{p^2}{p^2 - 1} \left(1 + \frac{1}{p} + \frac{\chi_3(p) - 1}{p^{v_p(n)/2+1}}\right) \prod_{\substack{p>3 \\ p \nmid n}} \left(1 - \frac{\chi_3(p)}{p}\right)^{-1} \\ &= \frac{\pi\sqrt{3}}{2} n^{1/2} \prod_p \frac{p^2 - 1}{p^2} \prod_p \left(1 - \frac{\chi_3(p)}{p}\right)^{-1} \prod_{p \mid n} \frac{p^2}{p^2 - 1} \left(1 + \frac{1}{p} + \frac{\chi_3(p) - 1}{p^{v_p(n)/2+1}}\right) \left(1 - \frac{\chi_3(p)}{p}\right) \\ &= \frac{\pi\sqrt{3}n^{1/2}}{2} \frac{1}{\zeta(2)} L(1, \chi_3) \prod_{p \mid n} \frac{p^2}{p^2 - 1} \left(1 + \frac{1}{p} + \frac{\chi_3(p) - 1}{p^{v_p(n)/2+1}}\right) \left(1 - \frac{\chi_3(p)}{p}\right) \\ &= \frac{\pi\sqrt{3}n^{1/2}}{2} \cdot \frac{6}{\pi^2} \cdot \frac{\pi\sqrt{3}}{9} \prod_{p \mid n} \frac{p^2}{p^2 - 1} \left(1 + \frac{1}{p} + \frac{\chi_3(p) - 1}{p^{v_p(n)/2+1}}\right) \left(1 - \frac{\chi_3(p)}{p}\right) \\ &= n^{1/2} \prod_{p \mid n} \frac{p^2}{p^2 - 1} \left(1 + \frac{1}{p} + \frac{\chi_3(p) - 1}{p^{v_p(n)/2+1}}\right) \left(1 - \frac{\chi_3(p)}{p}\right). \end{aligned}$$

We have that

$$\frac{p^2}{p^2 - 1} \left(1 + \frac{1}{p} + \frac{\chi_3(p) - 1}{p^{v_p(n)/2+1}}\right) \left(1 - \frac{\chi_3(p)}{p}\right) \geq 1$$

with equality if and only if $\chi_3(p) = 1$. It follows that

$$n = k^2 \text{ is not represented by } f \iff a_E(n) + a_H(n) = 0$$
$$\iff a_E(n) = n^{1/2} \iff \text{ for all primes } p \mid k, \chi_3(p) = 1.$$

Thus we have

$$\mathcal{E}_f(X) = \#\{n \leq x : n = k^2, \text{ and } p|k \implies p \equiv 1 \pmod 3\} \sim \frac{Cx^{1/2}}{\log^{1/4}(x)}$$

for some $C > 0$. So the upper bound of Proposition 4.4 is sharp up to log factors.

4.5. $n = 4$**, positive case.** Let $f_{/\mathbb{Z}}$ be a positive quaternary quadratic form. Then every sufficiently large $n \in \mathbb{Z}^+$ that is locally primitively represented – for all primes p there is $(x_1, x_2, x_3, x_4) \in \mathbb{Z}_p^4$ with at at least one x_i not divisible by p such that $f(x_1, x_2, x_3, x_4) = n$ – is primitively represented – there is $(x_1, x_2, x_3 x_4) \in \mathbb{Z}$ with $\gcd(x_1, x_2, x_3, x_4) = 1$ such that $f(x) = n$ [Ca, Thm. 11.1.6].

From this it follows easily that $\#\mathcal{E}_f(X) = O(\sqrt{X})$. Indeed, every representation of a squarefree integer is primitive, so $\mathcal{E}_f$ contains only finitely many squarefree exceptions, and there are $O(\sqrt{X})$ integers lying in the same rational square class as one of these squarefree exceptions. Thus after removing $O(\sqrt{X})$ integers, we may assume that $1 \leq n \leq X$ is such that for every integer m lying in the rational square class of n, if m is primitively locally represented then it is represented. We claim that n is represented. It certainly is if it is primitively locally represented. If not, there is a prime $p \mid n$ such that f $\mathbb{Z}_p$-represents $\frac{n}{p^2}$. For all primes $\ell \neq p$, we have $\frac{1}{p^2} \in \mathbb{Z}_\ell^{\times 2}$ so f $\mathbb{Z}_\ell$-represents $\frac{n}{p^2}$. Thus f locally represents $\frac{n}{p^2}$. If the representation is not locally primitive we can repeat the above argument, eventually getting that f primitively represents $m = \frac{n}{d^2}$ and thus also that f represents n.

4.6. $n \geq 5$**, positive case.** If $f_{/\mathbb{Z}}$ is a positive n-ary quadratic form with $n \geq 5$, then by Corollary 1.16, the form f is almost regular, so $\#\mathcal{E}_f(X) = O(1)$.

4.7. $n \geq 3$**, indefinite case.** In this case the result follows from work of Eichler, Kneser, Weil and Hsia, as recorded in [SP13, Thm. 5]. Indeed, if $f_{/\mathbb{Z}}$ is indefinite n-ary with $n \geq 4$ then f is regular and thus $\#\mathcal{E}_f(X) = 0$. If $f_{/\mathbb{Z}}$ is indefinite ternary then the elements of $\mathcal{E}_f$ lie in finitely many rational square classes and thus $\#\mathcal{E}_f(X) = O(\sqrt{X})$.

5. Lattice globalization

An n-ary quadratic form $f_{/\mathbb{Z}}$ has **signature** (r, s) if

$$f_{/\mathbb{R}} \cong x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2.$$

The following result amounts to telling us that for all $n \geq 3$, we can find an integral n-ary quadratic form that has prescribed signature, prescribed $\mathbb{Z}_p$-equivalence class at each of any finite set S of prime numbers, and has $\delta_p(f) = 1$ for all primes $p \notin S$. The underlying "globalization" construction is a well known one in the literature – see e.g. [Sh, Theorems 31.6 and 31.7] for a related result – but as we have not found a result exactly of the form we need, we supply a complete proof.

Theorem 5.1. *Let $n, r, s \in \mathbb{N}$ with $n \geq 3$ and $r + s = n$. Let S be a finite set of prime numbers, and let T be a subset of S. If $n = 3$, we suppose that $\#T$ is odd if $rs = 0$ and even otherwise. For $p \in S$, let f_p be an n-ary quadratic form defined over $\mathbb{Z}_p$ that is anisotropic if and only if $p \in T$. Then there is an n-ary quadratic form $f_{/\mathbb{Z}}$ such that:*

- *For all primes $p \in S$, we have $f_{/\mathbb{Z}_p} \cong f_p$.*
- *The form $f_{/\mathbb{R}}$ has signature (r, s).*
- *If $n = 3$, then for all primes $p \notin S$, we have that $f_{/\mathbb{Z}_p}$ is isotropic and maximal, hence universal.*
- *If $n \geq 4$, then for all primes $p \notin S$, we have that $f_{/\mathbb{Z}_p}$ is universal.*

Proof. Let $(f_\infty)_{/\mathbb{R}}$ be a quadratic form of signature (r,s).

Step 1a): Suppose $n=3$. By weak approximation [Ar, Thm. 8], there is $d \in \mathbb{Z}^+$ such that for all $p \in S$, we have $d \equiv \operatorname{disc} f_p \pmod{\mathbb{Q}_p^2}$ and $d>0$ if and only if s is even. For p a place of $\mathbb{Q}$ (i.e., a prime number or ∞) and $g_{/\mathbb{Q}_p}$ a ternary quadratic form, let $\epsilon_p(g) \in \{\pm 1\}$ be the Hasse invariant and let $c_p(g) = \epsilon(g)(\frac{-1,-\operatorname{disc} g}{\mathbb{Q}_p}) \in \{\pm 1\}$ be the Witt invariant. Recall that for a ternary quadratic form g over $\mathbb{Q}_p$, we have $c_p(g)=-1$ if and only if g is anisotropic. For p a place of $\mathbb{Q}$, put

$$c_p = \begin{cases} c_p(f_p) & \text{if } p \in S, \\ c_p(f_\infty) & \text{if } p = \infty, \\ 1 & \text{otherwise.} \end{cases} \tag{9}$$

By virtue of the parity condition imposed on T, we have $\prod_{p \leq \infty} c_p = 1$. Now for p a place of $\mathbb{Q}$, put

$$\epsilon_p \coloneqq c_p \left(\frac{-1,-d}{\mathbb{Q}_p} \right).$$

By (9) and the properties of the Hilbert symbol, we have that $\epsilon_p = 1$ except for finitely many p and $\prod_{p \leq \infty} \epsilon_p = 1$. Since two ternary quadratic forms over $\mathbb{Q}_p$ with the same discriminant square class and Witt invariants are isometric, by [Se, Prop. 7], there is a quadratic form $q_{/\mathbb{Q}}$ of signature (r,s) such that $\operatorname{disc} q \equiv d \pmod{\mathbb{Q}^2}$, $q_{/\mathbb{Q}_p} \cong q_p$ for all $p \in S$ and q is isotropic at all prime numbers $p \notin S$.

Step 1b): Suppose $n \geq 4$. Fix a finite odd prime $p_0 \notin S$. For p a place of $\mathbb{Q}$, put

$$\epsilon_p = \begin{cases} \epsilon_p(f_p) & \text{if } p \in S \cup \{\infty\}, \\ 1 & \text{if } p \notin S \cup \{p_0, \infty\}, \end{cases} \tag{10}$$

and take $\epsilon_{p_0} \in \{\pm 1\}$ to be so that

$$\prod_{p \leq \infty} \epsilon_p = 1.$$

Now let $(q_{p_0})_{/\mathbb{Q}_{p_0}}$ be an isotropic n-ary quadratic form with $\epsilon_{p_0}(q_{p_0}) = \epsilon_{p_0}$. (Since

$$\epsilon_{p_0}(t_1^2 + \ldots + t_n^2) = 1$$

and

$$\epsilon_{p_0}(t_1^2 + \ldots + t_{n-2}^2 + r t_{n-1}^2 + p_0 t_n^2) = -1,$$

this is certainly possible.) By weak approximation, there is $d \in \mathbb{Z}^+$ such that for all $p \in S \cup \{p_0\}$, we have $d \equiv \operatorname{disc} f_p \pmod{\mathbb{Q}_p^2}$ and $d>0$ if and only if s is even. Applying [Se, Prop. 7] there is an n-ary quadratic form $q_{/\mathbb{Q}}$ of signature (r,s) such that $q_{/\mathbb{Q}_p} \cong q_p$ for all $p \in S \cup \{p_0\}$.

Step 2: We view the quadratic form $q_{/\mathbb{Q}}$ constructed in Step 1 as a quadratic space $V_{/\mathbb{Q}}$. For each $p \in S$, we may view f_p as a $\mathbb{Z}_p$-lattice L_p in the quadratic space $V \otimes \mathbb{Q}_p$. Let M be any maximal $\mathbb{Z}$-lattice in V. By [Ge, Thm. 9.4], there is a lattice L in V such that for a prime number p, we have

$$L \otimes \mathbb{Z}_p = \begin{cases} L_p & \text{if } p \in S, \\ M_p & \text{otherwise.} \end{cases}$$

The quadratic form $f_{/\mathbb{Z}}$ corresponding to L satisfies the desired properties: If $n=3$, then for all $p \notin S$, the form $f_{/\mathbb{Z}_p}$ is iostropic and maximal, hence universal, while if $n \geq 4$ then for all $p \notin S$ the form $f_{/\mathbb{Z}_p}$ is maximal, hence ADC, hence universal. □

6. Proofs of the remaining theorems

6.1. **Proof of Theorem 1.6b).** Let $f_{/\mathbb{Z}}$ be a primitive isotropic binary quadratic form of Discriminant Δ. By (2) we have $\delta(f) = \prod_p \delta_p(f)$. For $p>2$, Case 1 of §2.2.1 and (4) computes $\delta_p(f)$, while Proposition 2.6 computes $\delta_2(f)$.

Remark 6.1. Our local analysis also shows that for a primitive isotropic binary quadratic form $f_{/\mathbb{Z}}$, the set $D_{f,\mathrm{loc}}$ is a union of congruence classes modulo Δ, so $\delta(f) = \frac{N(\Delta)}{\Delta}$, where $N(\Delta)$ is the number of congruence classes modulo Δ that are locally represented by f.

6.2. **Proof of Theorem 1.7.** Let $n \geq 3$ and let $f_{/\mathbb{Z}}$ be an n-ary quadratic form. We have

$$\delta(f) = \prod_p \delta_p(f).$$

Each $\delta_p(f)$ is positive. Moreover, since $n \geq 3$, for each odd prime $p \nmid \operatorname{disc}(f)$ we have that $f_{/\mathbb{Z}_p}$ is universal, so $\delta_p(f) = 1$. It follows that $\delta(f) > 0$.

6.3. **Proof of Theorem 1.8.** Let $f_{/\mathbb{Z}}$ be an anisotropic ternary form. By Hasse-Minkowski, f is anisotropic at a nonempty, finite set of places of even cardinality hence over $\mathbb{Z}_p$ for at least one prime p. As seen in §2, f then does not represent any element in the $\mathbb{Q}_p$-square class of $-\operatorname{disc} f$, which forces $\delta_p(f) < 1$ and thus $\delta(f) = \prod_p \delta_p(f) < 1$.

6.4. **Proof of Theorem 1.11.** Let $n \geq 3$, and let $f_{/\mathbb{Z}}$ be an n-ary quadratic form. If $n = 3$ we assume that f isotropic. For every prime p, the form $f_{/\mathbb{Q}_p}$ is universal. Thus f is locally universal if and only if it is locally ADC. Moreover, since every $\mathbb{Z}_p$-square class has positive measure, these conditions are also equivalent to $\delta_p(f) = 1$ for all p and thus, since $\delta(f) = \prod_p \delta_p(f)$, to $\delta(f) = 1$. This proves Theorem 1.11a). Part b) follows from part a) and Theorem 1.1: if f is maximal, it is locally maximal and thus locally ADC.

6.5. **Proof of Theorem 1.12.** Let $n \geq 1$ and let $f_{/\mathbb{Z}}$ be an n-ary quadratic form. By (4) we have $\delta_p \in (0, 1] \cap \mathbb{Q}$ for all primes p. If $n \geq 3$ or $n = 2$ but $f_{/\mathbb{Z}}$ is isotropic, then $\delta_p(f) = 1$ for all but finitely many primes p; hence, $\delta(f) = \prod_p \delta_p(f)$ is rational. The remaining case is that $n \leq 2$ and f is anisotropic, and in this case we've already seen that $\delta(f) = 0$.

6.6. **Proof of Theorem 1.13.**

Lemma 6.2. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with values in $(0, 1]$. We suppose:*

(i) $a_n \to 0$,
(ii) $\sum_{n=1}^{\infty} a_n = \infty$.

Then: for all $0 \leq \alpha < \beta \leq 1$, there are positive integers $N, n_1, \ldots, n_N$ such that $\prod_{n=1}^{N}(1-a_{n_i}) \in (\alpha, \beta)$.

Proof. Put $r := \frac{\alpha}{\beta}$. Choose $K \in \mathbb{Z}^+$ such that for all $n > K$ we have $1 - a_n > r$. We claim that there is $L \in \mathbb{Z}^+$ such that $\prod_{i=1}^{L}(1 - a_{K+i}) \in (\alpha, \beta)$. Indeed, since $\sum_{n=1}^{\infty} a_n = \infty$ we have $\prod_{n=1}^{\infty}(1 - a_n) = 0$, so certainly $\prod_{i=1}^{L}(1 - a_{K+i}) < \beta$ for all sufficiently large L. Let L be the smallest such positive integer. Then, since $\prod_{i=1}^{L-1}(1 - a_{K+i}) > \beta$, we have

$$\prod_{i=1}^{L}(1 - a_{K+i}) > \beta(1 - a_{k+L}) > r\beta = \alpha. \qquad \square$$

Now we prove Theorem 1.13. For $n \in \mathbb{Z}^+$, let p_n be the nth odd prime number, and put $a_n := \frac{1}{2p_n+2}$. This sequence satisfies (i) and (ii) of Lemma 6.2. Thus for all $0 \leq \alpha < \beta \leq 1$ there are $N, n_1, \ldots, n_N$ such that $\prod_{i=1}^{N}(1 - \frac{1}{2p_{n_i}+2}) \in (\alpha, \beta)$.

Let us first treat the case in which $n \geq 4$. Then we put $S := \{p_{n_1}, \ldots, p_{n_N}\}$. By the local analysis of §2, for every prime $p > 2$ there is an (anisotropic) quadratic form $(f_p)_{/\mathbb{Z}_p}$ with $\delta_p(f_p) = 1 - \frac{1}{2p+2}$ that represents at least one of 1 and r, hence is primitive. Applying Theorem 5.1 we get an n-ary form $f_{/\mathbb{Z}}$ of signature (r, s) such that $f_{/\mathbb{Z}_p} \cong f_p$ for all $p \in S$ and such that $f_{/\mathbb{Z}_p}$ is universal – hence primitive, for all $p \notin S$. This form f is locally primitive – hence primitive – and has

$$\delta(f) = \prod_{p \in S} \delta_p(f) = \prod_{p \in S}\left(1 - \frac{1}{2p+2}\right) \in (\alpha, \beta).$$

Now suppose $n = 3$. We want to apply the above argument with $S = T$, but there is now a parity requirement on T that may not be satisfied. If it is not, we choose $p_{n_{N+1}}$ large enough so that

$$\prod_{i=1}^{N+1}\left(1-\frac{1}{2p_{n_i}+2}\right) \in (\alpha,\beta)$$

and put $S = T = \{p_{n_1}, \ldots, p_{n_{N+1}}\}$. The remainder of the argument is the same.

6.7. **Proof of Theorem 1.14.** Put

$$\Delta_2 \coloneqq \left\{\frac{1}{2},\ \frac{5}{6},\ \frac{11}{12}, 1\right\},$$

and for an odd prime p put

$$\Delta_p \coloneqq \left\{\frac{1}{2},\ \frac{p+2}{2p+2},\ \frac{2p+1}{2p+2}, 1\right\}.$$

By Theorem 1.20 and the material of §2, we have $\delta(f) = \prod_p \delta_p(f)$ with $\delta_p(f) \in \Delta_p$ for all p and $\delta_p = 1$ for all but finitely many p.

Step 1: If f is isotropic and ADC then it is universal and $\delta(f) = 1$. So we may assume that f is anisotropic and thus is anisotropic at $k \geq 1$ primes p. For each of these primes, we have that $v_2(\delta_p(f)) < 0$ and thus $v_2(\delta(f)) \leq -k$. It follows that the proportion of attained values of δ within $\mathbb{Q} \cap [0, 1]$ approaches 0 as $k \to \infty$ so it suffices to deal with a fixed number k of anisotropic primes.

Step 2: For each fixed $k \in \mathbb{Z}^+$, let $\mathcal{D}_k$ be the set of rational numbers $\delta = \prod_p \delta_p$ of the above form such that $\delta_p \neq 1$ for at most k primes. We claim that $\mathcal{D}_k$ is a compact subset of $[0, 1]$. First we observe that every sequence in $\mathcal{D}_1$ has a subsequence that is either constant, convergent to 1 or convergent to $\frac{1}{2}$, so $\mathcal{D}_1$ is a sequentially compact metric space, thus compact. Next we observe that $\mathcal{D}_k$ is the image of $\prod_{i=1}^{k} \mathcal{D}_1$ under the continuous map $(x_1, \ldots, x_k) \mapsto x_1 \cdots x_k$ so is compact.

Step 3: We show that any compact subset $K \subset \mathbb{Q} \cap [0, 1]$ contains only 0% of $\mathbb{Q} \cap [0, 1]$ in the sense of height. We do so by contraposition: For each $n \in \mathbb{Z}^+$, let K_n be the subset of K consisting of rational numbers with reduced deminator n. Since there are $\gg N^2$ elements $x \in [0, 1] \cap \mathbb{Q}$ with numerator and denominator at most N, if K does not contain 0% of the rational numbers, there is $c > 0$ and infinitely many $N \in \mathbb{Z}^+$ such that the number of elements of K with denominator at most N is at least cN^2, and thus there is an infinite subset $J \subset \mathbb{Z}^+$ such that

$$\forall n \in J,\ \#K_n > cn:$$

for if not, then the number of rationals in K with reduced denominator at most N would be at most $\sum_{n=1}^{N} cn + O(1)$, which is smaller than cN^2 for large N. Now we thicken each K_n to a set U_n by surrounding each point of K_n with the intersection with $[0, 1]$ of an open interval of radius $\frac{1}{2n}$. Since distinct elements of K_n are at least $\frac{1}{n}$ apart, these intervals are pairwise disjoint, and thus for each $n \in J$ the measure of U_n is at least $\#K_n \cdot \frac{1}{2n} \geq \frac{c}{2}$. Let U be the set of reals that belong to U_n for infinitely many n. Then the measure of U is at least $\frac{c}{2}$ [H, p. 40]. Every element of U is the limit of a sequence in K, but K is compact, so $K \supset U$ and thus K has positive measure. So K cannot be contained in $\mathbb{Q}$.

6.8. **Remarks on 2-adic properties of the density.** We do not know whether the "locally ADC" condition in Theorem 1.14 can be removed. One the one hand, we have not completed the classification of densities of quadratic forms over $\mathbb{Z}_2$. Using our present methods this would be tedious to do without computer assistance. But this is not really the crux of the matter: consider for instance the problem of finding all densities of ternary integral quadratic forms f such that $f_{/\mathbb{Z}_2}$ is universal. In this case, we know everything that we need to know on the quadratic forms side, but we cannot solve the elementary number-theoretic problem of determining which rational numbers arise as $\prod_p \delta_p$ for the known possible values of δ_p. The proof of Theorem 1.14 does not adapt to this case, because it exploits

2-adic properties of $\delta_p(f)$ and $\delta(f)$. There is a curious phenomenon here: loosely speaking, "$v_2(\delta(f))$ wants to be negative but does not need to be negative." We make this precise in the following results.

Theorem 6.3. *Let $f_{/\mathbb{Z}}$ be a positive definite ternary quadratic form. Let U be the set of primes $p \equiv 3 \pmod 4$ such that $f_{/\mathbb{Z}_p}$ is isotropic. Suppose that $f_{/\mathbb{Z}_p}$ is ADC for $p = 2$ and for all $p \in U$. Then*

$$v_2(\delta(f)) < 0.$$

Proof. As above, let S be the set of prime numbers p such that $f_{/\mathbb{Z}_p}$ is not both anisotropic and maximal, and let T be the subset of S of anisotropic places; $\#T$ is odd and thus positive. Then

$$\delta(D_f) = \prod_{p \in S} \delta_p(f).$$

We claim that $v_2(\delta_p(f)) \leq 0$ for all $p \in S$ and $v_2(\delta_p(f)) < 0$ for all $p \in T$: certainly, this suffices.

- We have $v_2(\delta_2(f)) \leq 0$ by Theorem 2.7; henceforth we suppose $p > 2$.
- Suppose $p \in S \setminus T$, so $f_{/\mathbb{Z}_p}$ is isotropic. By hypothesis, if $p \equiv 3 \pmod 4$, then $f_{/\mathbb{Z}_p}$ is ADC, so again Theorem 2.7 gives $v_2(\delta_p(f)) \leq 0$. Now suppose $p \equiv 1 \pmod 4$. Let X be a set of representatives of the $\mathbb{Z}_p$-square classes that f does not represent. Thus

$$\delta_p(f) = 1 - \sum_{x \in X} \frac{p-1}{2p^{v(x)+1}}.$$

  Since $p \equiv 1 \pmod 4$, each term has positive 2-adic valuation, and thus $\delta_p(f) = 0$.
- Suppose $p \in T$, so $f_{/\mathbb{Z}_p}$ is anisotropic. Let $g_{/\mathbb{Z}_p}$ correspond to a maximal lattice containing f. Then arguing as in the previous case we get

$$\delta_p(f) = \delta_p(g) - \sum_{x \in X} \frac{p-1}{2p^{v(x)+1}}.$$

  Since each term of the sum has non-negative 2-adic valuation and $v_2(\delta_p(g)) < 0$, we get $v_2(\delta_p(f)) = v_2(\delta_p(g)) < 0$. □

Proposition 6.4. *For every $k \in \mathbb{N}$ there is a positive definite ternary quadratic form $f_{/\mathbb{Z}}$ such that $v_2(\delta(f)) \geq k$.*

Proof. By Dirichlet's Theorem, there is a prime number p such that $p \equiv -1 \pmod{2^{k+2}}$. Let $S = \{2, p\}$, $T = \{2\}$ and

$$f_2 = x^2 + y^2 + z^2,$$
$$f_p = x^2 + py^2 - pz^2.$$

The form $(f_2)_{/\mathbb{Z}_2}$ is anisotropic, and the form $(f_p)_{/\mathbb{Z}_p}$ is isotropic. Thus Theorem 5.1 applies, and there is a positive definite ternary $f_{/\mathbb{Z}}$ such that $f_{/\mathbb{Z}_2} \cong f_2$, $f_{\mathbb{Z}_p} \cong f_p$ and $f_{/\mathbb{Z}_\ell}$ is isotropic and maximal for all $\ell \neq 2, p$. So

$$\delta(f) = \delta_2(f)\delta_p(f) = \frac{5}{6} \cdot \frac{p+1}{2p}.$$

By our choice of p, we have $2^{k+2} \mid p + 1$, and the result follows. □

References

[Al33] A.A. Albert, *The integers represented by sets of ternary quadratic forms.* Amer. J. Math. 55 (1933), 274–292.

[Ar] E. Artin, *Algebraic numbers and algebraic functions.* Gordon and Breach Science Publishers, New York-London-Paris 1967.

[Be12] P. Bernays, *Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante*, Dissertation, Göttingen, 1912, available at `http://www.math.uni-bielefeld.de/~rehmann/DML/`

[Bl04] V. Blomer, *Uniform bounds for Fourier coefficients of theta-series with arithmetic applications.* Acta Arith. 114 (2004), 1–21.

[BMO11] D. Brink, P. Moree and R. Osburn, *Principal forms $X^2 + nY^2$ representing many integers.* Abh. Math. Semin. Univ. Hambg. 81 (2011), 129–139.

[BO08] J. Bochnak and B.-K. Oh, *Almost regular quaternary quadratic forms.* Ann. Inst. Fourier (Grenoble) 58 (2008), 1499–1549.

[Ca] J.W.S. Cassels, *Rational quadratic forms.* London Mathematical Society Monographs, 13. Academic Press, Inc., London-New York, 1978.

[Cl12] P.L. Clark, *Euclidean quadratic forms and ADC forms: I* Acta Arith. 154 (2012), 137–159.

[CJ14] P.L. Clark and W.C. Jagy, *Euclidean quadratic forms and ADC forms II: integral forms.* Acta Arith. 164 (2014), 265–308.

[CO04] W.K. Chan and B.-K. Oh, *Positive ternary quadratic forms with finitely many exceptions.* Proc. Amer. Math. Soc. 132 (2004), 1567–1573.

[CS91] J. Conway and N. Sloane, *Lattices with few distances.* J. Number Theory 39 (1991), 75–90.

[DW17] G. Doyle and K.S. Williams, *A positive-definite ternary quadratic form does not represent all positive integers.* Integers 17 (2017), Paper No. A41, 19 pages.

[Fo96] O.M. Fomenko, *Distribution of values of Fourier coefficients of modular forms of weight 1.* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 226 (1996), Anal. Teor. Chisel i Teor. Funktsiĭ. 13, 196–227, 240; translation in. J. Math. Sci. (New York) 89 (1998), no. 1, 1050–1071

[Ga] C.F. Gauss, *Disquisitiones arithmeticae.* Translated and with a preface by Arthur A. Clarke. Revised by William C. Waterhouse, Cornelius Greither and A. W. Grootendorst and with a preface by Waterhouse. Springer-Verlag, New York, 1986.

[Ge] L.J. Gerstein, *Basic quadratic forms.* Graduate Studies in Mathematics, 90. American Mathematical Society, Providence, RI, 2008.

[H] P.R. Halmos, *Measure Theory.* D. Van Nostrand Company, Inc., New York, N. Y., 1950.

[Ha04] J. Hanke, *Local densities and explicit bounds for representability by a quadratric form.* Duke Math. J. 124 (2004), 351–388.

[JKS97] W.C. Jagy, I. Kaplansky and A. Schiemann, *There are 913 regular ternary forms.* Mathematika 44 (1997), 332–341.

[JP39] B. W. Jones, G. Pall, *Regular and semi-regular positive ternary quadratic forms*, Acta Math. 70 (1939), 165–191.

[L] T.-Y. Lam, *Introduction to quadratic forms over fields.* Graduate Studies in Mathematics, 67. American Mathematical Society, Providence, RI, 2005.

[La08] E. Landau, *Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate*. Arch. Math. Phys. 13 (1908), 305–312.

[LO14] R.J. Lemke Oliver, *Representation by ternary quadratic forms.* Bull. Lond. Math. Soc. 46 (2014), 1237—1247.

[MO] `https://mathoverflow.net/questions/75399`

[MO06] P. Moree and R. Osburn, *Two-dimensional lattices with few distances.* Enseign. Math. (2) 52 (2006), 361–380.

[Od77] R.W.K. Odoni, *A new equidistribution property of norms of ideals in given classes.* Acta Arithmetica 33 (1977), 53–63.

[RP46] A.E. Ross and G. Pall, *An extension of a problem of Kloosterman.* Amer. J. Math. 68 (1946), 59–65.

[Se] J.-P. Serre, *A course in arithmetic.* Translated from the French. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973.

[Sh] G. Shimura, *Arithmetic of quadratic forms.* Springer Monographs in Mathematics. Springer, New York, 2010.

[SP80] R. Schulze-Pillot, *Darstellung durch Spinorgeschlechter ternärer quadratischer Formen.* J. Number Theory 12 (1980), no. 4, 529–540.

[SP13] R. Schulze-Pillot, *Representation of quadratic forms by integral quadratic forms.* Quadratic and higher degree forms, 233–253, Dev. Math., 31, Springer, New York, 2013.

[Wa] G. L. Watson, *Integral quadratic forms.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 51. Cambridge University Press, New York, 1960.

[Ya98] T. Yang, *An explicit formula for local densities of quadratic forms.* J. Number Theory, 72 (1998), 309–356.