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Clusters of primes with square-free translates

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Abstract. Let $\mathcal{R}$ be a finite set of integers satisfying appropriate local conditions. We show the existence of long clusters of primes p in bounded length intervals with $p - b$ squarefree for all $b \in \mathcal{R}$. Moreover, we can enforce that the primes p in our cluster satisfy any one of the following conditions: (1) p lies in a short interval $[N, N + N^{\frac{7}{12}+\varepsilon}]$, (2) p belongs to a given inhomogeneous Beatty sequence, (3) with $c \in (\frac{8}{9}, 1)$ fixed, p^c lies in a prescribed interval mod 1 of length $p^{-1+c+\varepsilon}$.

1. Introduction

Recent work on small gaps between primes owes a considerable debt to the innovative use of the Selberg sieve by Goldston, Pintz, and Yildirim [8]. This paper contains the result, for the sequence of primes $p_1, p_2, \ldots$,

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0. \tag{1.1}$$

By adapting the method, Zhang [20] achieved the breakthrough result

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < \infty.$$

Not long afterwards, Maynard [11] refined the sieve weights of Goldston, Pintz, and Yildirim to obtain the stronger result, for $t = 2, 3, \ldots$

$$\liminf_{n\to\infty} (p_{n+t-1} - p_n) \ll t^3 e^{4t}. \tag{1.2}$$

The implied constant is absolute. Similar results were obtained at the same time by Tao (unpublished). Tao's use of weights is available in the paper [16] by the Polymath group; for some problems, this is a more convenient approach than that of Maynard [11]. Polymath [15] also refined the work of Zhang [20] to obtain new equidistribution estimates for primes in arithmetic progressions. When combined

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with techniques in [16], the outcome (see [16]) is a set of results that are explicit for the left-hand side of (1.2), for small t, and give $O\left(t\exp\left(\left(4-\frac{28}{157}\right)t\right)\right)$ for $t\geq 2$ in place of the bound in (1.2). The latter result has been sharpened further by Baker and Irving [2]. In a different direction, Ford, Green, Konyagin, Maynard, and Tao [7] have used the Maynard-Tao method in giving a breakthrough result on *large* gaps between primes.

It is natural to ask whether a given infinite sequence of primes $\mathcal{B}=\{p_1',p_2',\ldots\}$ satisfies a bound analogous to (1.2), say

$$\liminf_{n\to\infty}\,(p'_{n+t-1}-p'_n)\ll F(\mathcal{B},t)\quad (t=2,3,\ldots). \tag{1.3}$$

In the present paper we answer affirmatively a question of this kind raised by Benatar [5]. Let b_1 be a fixed nonzero integer and

$$\mathcal{B}=\{p : p \text{ prime}\ ,\ p-b_1 \text{ is square-free}\}.$$

Does (1.3) hold for $t=2$? (Benatar was able to obtain the analogue of (1.1) for primes in $\mathcal{B}$.) It is of some interest to consider more generally a *set* of translates

$$\mathcal{R}=\{b_1,\ldots,b_s\} \tag{1.4}$$

and the set

$$\mathcal{B}(\mathcal{R})=\{p : p \text{ prime},\ p-b \text{ is squarefree for all } b\in\mathcal{R}\}. \tag{1.5}$$

There are simple local conditions that $\mathcal{R}$ must satisfy.

Definition. A set $\{b_1,\ldots,b_s\}$ of nonzero integers is *reasonable* if for every prime p there is an integer v, $p\nmid v$, with

$$b_\ell\not\equiv v \pmod{p^2}\quad (\ell=1,\ldots,s).$$

A little thought shows that, if there are infinitely many primes p with $p-b_1,\ldots,p-b_s$ all square-free, then $\{b_1,\ldots,b_s\}$ is a reasonable set.

Theorem 1. *Let $t>1$ and $\varepsilon>0$. Let $\mathcal{R}$ be a reasonable set of cardinality s and define $\mathcal{B}(\mathcal{R})$ by (1.5). The sequence $p_1',p_2',\ldots$ of primes in $\mathcal{B}(\mathcal{R})$ satisfies*

$$\liminf_{n\to\infty}\,(p'_{n+t-1}-p'_n)\leq \exp(C_1(\varepsilon)s\exp((4+\varepsilon)t)).$$

From now on, let $\mathcal{R}$ be a fixed reasonable set of cardinality s, given by (1.4). We now pursue the possibility of finding clusters of primes p for which $p-b$ is squarefree for all $b\in\mathcal{R}$, *and* p is chosen from a given subset $\mathcal{A}$ of $[N,2N]$ for a sufficiently large positive integer N. This is in the spirit of the papers of Maynard [12] and Baker and Zhao [3], which contain overlapping theorems of the following kind: *Given sufficient arithmetic regularity of $\mathcal{A}\subset[N,2N]$, there is a set $\mathcal{S}$ of t primes in $\mathcal{A}$ with diameter*

$$D(\mathcal{S}):=\max_{n\in\mathcal{S}}n-\min_{n\in\mathcal{S}}n\ll F(t)\quad (t=2,3,\ldots). \tag{1.6}$$

Here F depends on certain properties of $\mathcal{A}$. Theorems 2, 3, and 4 are of this kind, for three different choices of $\mathcal{A}$, with the additional requirement that $p-b$ *is squarefree for all* p *in* $\mathcal{S}$ *and* b *in* $\mathcal{R}$.

Our first example $\mathcal{A}$ is

$$\mathcal{A}_1(\phi) = \mathbb{Z} \cap [N, N+N^{\phi}],$$

where ϕ is a constant in $(7/12, 1]$. The existence of a set $\mathcal{S}$ of t primes in $\mathcal{A}_1(\phi)$ satisfying (1.6) is due to Maynard [12], with $F(t)$ of the form $\exp(K(\phi)t)$.

Our second example is suggested by work of Baker and Zhao [3]. Let $\lfloor w \rfloor$ denote the integer part of w. A *Beatty sequence* is a sequence

$$\lfloor \alpha m + \beta \rfloor, \ m = 1, 2, \ldots$$

where α is a given irrational number, $\alpha > 1$ and β is a given real number. We write $\mathcal{A}_2(\alpha, \beta)$ for the intersection of this sequence with $[N, 2N]$. The existence of a set $\mathcal{S}$ of t primes in $\mathcal{A}(\alpha, \beta)$ is shown in [3], for a family of values of N depending on α, with

$$F(t) = (t + \log \alpha) \exp(7.743t).$$

Let c be a constant in $(8/9, 1)$. A third example, not previously considered in connection with clusters of primes, is

$$\mathcal{A}_3(c, \varepsilon) = \{n \in [N, 2N) : n^c \in I \pmod 1\},$$

where $\varepsilon > 0$ and I is an interval of length

$$|I| = N^{-1+c+\varepsilon}. \tag{1.7}$$

A corollary of Theorem 4 below is that $\mathcal{A}_3(c, \varepsilon)$ contains a set $\mathcal{S}$ of t primes whose diameter is bounded as in (1.6). The problem of finding, or enumerating asymptotically, primes in sets similar to $\mathcal{A}_3(c, \varepsilon)$, but with I of more general length, has been studied by Balog [4] and others. We note a connection with the problem of finding primes of the form $[n^C]$. See e.g. Rivat and Wu [17], where $1 < C < 243/205$. Let $\gamma = 1/C$. The number of primes of the form $[n^C]$, $n \leq x$, is given by

$$\sum_{p \leq x} (\lfloor -p^{\gamma}] - [-(p+1)^{\gamma}]) + O(1). \tag{1.8}$$

The sum in (1.8) counts the number of $p \leq x$ with $-p^{\gamma} \in J_p \pmod 1$, where $J_p = (1 - \ell_p, 1)$ with $\ell_p \sim \gamma p^{\gamma - 1}$.

In $[N, 2N]$, there cannot be two primes $p < p_1$ with $p_1 - p = O(1)$ and $p_1^c - p^c$ smaller $\pmod 1$ than N^{c-1}. For

$$p_1^c - p^c \geq c p_1^{c-1}(p_1 - p) \geq 2c(2N)^{c-1}.$$

This explains the choice of exponent $c - 1 + \varepsilon$ in (1.7).

We now state results about clusters of primes with square-free translates in $\mathcal{A}_1(\phi)$, $A_2(\alpha, \beta)$ and $\mathcal{A}_3(c, \varepsilon)$. We write $C_2, C_3, \ldots$ for certain absolute constants.

Theorem 2. *Let $t > 1$, $7/12 < \phi < 1$. Let*

$$\psi = \begin{cases} \phi - 11/20 - \varepsilon & (7/12 < \phi < 3/5) \\ \phi - 1/2 - \varepsilon & (\phi \geq 3/5). \end{cases}$$

For sufficiently large N, there is a set $\mathcal{S}$ of t primes in $\mathcal{A}_1(\phi)$ such that

$$p - b \text{ is squarefree} \quad (p \in \mathcal{S},\ b \in \mathcal{R}) \tag{1.9}$$

and

$$D(\mathcal{S}) < \exp\left(C_2 s \exp\left(\frac{2t}{\psi}\right)\right).$$

Theorem 3. *Let $t > 1$. Let α be an irrational number, $\alpha > 1$ and let β be real. Let v be a sufficiently large integer such that*

$$\left|\alpha - \frac{u}{v}\right| < \frac{1}{v^2} \quad \text{for some } u \text{ with } (u, v) = 1.$$

For sufficiently large $N = v^2$, there is a set $\mathcal{S}$ of t primes in $\mathcal{A}_2(\alpha, \beta)$ satisfying (1.9) *and*

$$D(\mathcal{S}) < \exp(C_3 \alpha s \exp(7.743t)). \tag{1.10}$$

Theorem 4. *Let $t > 1$. Let $8/9 < c < 1$ and let β be real. Let $0 < \psi < (9c - 8)/6$ and $\varepsilon > 0$. Let $I = [\beta, \beta + N^{-1+c+\varepsilon}]$. For sufficiently large N, there is a set $\mathcal{S}$ of t primes in $\mathcal{A}_3(c, \varepsilon)$ such that* (1.9) *holds, and*

$$D(\mathcal{S}) < \exp\left(C_4 st \exp\left(\frac{2t}{\psi}\right)\right). \tag{1.11}$$

We shall deduce these theorems from a general result of the same kind concerning a subset $\mathcal{A}$ of $[N, 2N]$ satisfying arithmetic regularity conditions (Theorem 5). In Section 2 we state Theorem 5 and explain the strategy of proof. Section 3 contains the proof of Theorem 5. In subsequent sections we deduce Theorems 1, 2, 3 and 4 from Theorem 5.

Note that Theorems 3 and 4 lead to conclusions of the form (1.3) both for $\mathcal{B}$ a Beatty sequence and for

$$\mathcal{B} = \{p : p \text{ prime}, \{p^c - \beta\} < p^{-1+c+\varepsilon}\}$$

$\left(\beta \text{ real}, \frac{8}{9} < c < 1\right)$.

2. A general theorem on clusters of primes with square-free translates.

In the present section we suppose that t is fixed and N is sufficiently large, and write $\mathcal{L} = \log N$,

$$D_0 = \frac{\log N}{\log\log N}.$$

We denote by $\tau(n)$ and $\tau_k(n)$ the usual divisor functions. Let ε be a sufficiently small positive number. Let $X(E;\ldots)$ denote the indicator function of a set E. Let

$$P(z) = \prod_{p<z} p.$$

A set of integers $\mathcal{H}_k = \{h_1, \ldots, h_k\}$, $0 \le h_1 < \cdots < h_k$ is said to be *admissible* if for every prime p, $\mathcal{H}_k \pmod p$ does not cover all residue classes $\pmod p$. An admissible set $\mathcal{H}_k$ is said to be *compatible* with $\mathcal{R}$ if

$$h_m \equiv 0 \pmod{P^2} \quad (m = 1, \ldots, k) \tag{2.1}$$

where

$$P := P((s+1)k+1) \tag{2.2}$$

and further

$$h_i - h_j + b \ne 0 \quad (i \ne j,\ b \in \mathcal{R}). \tag{2.3}$$

In the applications in Sections 4–6, it is not difficult to produce sets compatible with $\mathcal{R}$ and which (in the case of Theorem 3) possess another useful property.

A few remarks will clarify the purpose of compatibility. For brevity, we say that $n - \mathcal{R}$ is *square-free* if $n - b$ is square-free for every $b \in \mathcal{R}$, and that $\mathcal{C} - \mathcal{R}$ is *square-free* if $n - \mathcal{R}$ is square-free for all $n \in \mathcal{C}$. Once we have fixed a suitable set $\mathcal{A}$ in $[N, 2N]$ and $t \in \mathbb{N}$, we show that for *many* n in $\mathcal{A}$, at least t of $n+h_1, \ldots, n+h_k$ are primes in $\mathcal{A}$. (We need k large, as a function of t.) Compatibility of $\mathcal{H}$ with $\mathcal{R}$ is now needed to show that only a *few* n in $\mathcal{A}$ have $n + h - b$ not squarefree for some $h \in \mathcal{H}_k$ and $b \in \mathcal{B}$. Select a 'satisfactory' n and let $\mathcal{S}$ be a set of t primes in $\{n + h_1, \ldots, n + h_k\}$; then $D(\mathcal{S}) \le h_k - h_1$ and $\mathcal{S} - \mathcal{R}$ is square-free.

In the proof of Theorem 5, we use a smooth function F supported on

$$\mathcal{E}_k := \left\{ (x_1, \ldots, x_k) \in [0,1]^k : \sum_{j=1}^{k} x_j \le 1 \right\}$$

with a special property. Let

$$I_k(F) := \int_0^1 \cdots \int_0^1 F(t_1, \ldots, t_k)^2 dt_1 \ldots dt_k,$$

$$J_k^{(m)}(F) = \int_0^1 \cdots \int_0^1 \left(\int_0^1 F(t_1, \ldots, d_k)^2 dt_m \right) dt_1 \ldots dt_{-1} dt_{m+1} \ldots dt_k$$

for $1 \le m \le k$. We choose F so that

$$\sum_{m=1}^{k} J_k^{(m)}(F) > (\log k - C_5) I_k(F) > 0; \tag{2.4}$$

this is possible by [16, Theorem 3.9].

Let $\mathbb{P}$ denote the set of prime numbers.

Theorem 5. *Let $t > 1$. Let $\mathcal{H}_k$ be compatible with $\mathcal{R}$. Let $N \in \mathbb{N}$, $N > C_0(\mathcal{R}, \mathcal{H}_k)$. Let $N^{1/2}\mathcal{L}^{18k} \le M \le N$ and let $\mathcal{A} \subset [N, N+M] \cap \mathbb{Z}$. Let θ be a constant, $0 < \theta < 3/4$. Let Y be a positive number,*

$$N^{1/4} \max(N^\theta, \mathcal{L}^{9k} M^{1/2}) \ll Y \ll M. \tag{2.5}$$

Let

$$V(q) := \max_a \left| \sum_{n \equiv a \pmod q} X(\mathcal{A}; n) - \frac{Y}{q} \right|.$$

Suppose that, for

$$1 \le d \le (MY^{-1})^4 \max(\mathcal{L}^{36k}, N^{4\theta} M^{-2}), \tag{2.6}$$

we have

$$\sum_{\substack{q \le N^\theta \\ (q,d)=1}} \mu^2(q) \tau_{3k}(q) V(dq) \ll Y \mathcal{L}^{-k-\varepsilon} d^{-1}. \tag{2.7}$$

Suppose there is a function $\rho(n) : [N, 2N] \cap \mathbb{Z} \to \mathbb{R}$ such that

$$X(\mathbb{P}; n) \ge \rho(n) \quad (N \le n \le 2N) \tag{2.8}$$

and positive numbers $Y_1, \ldots, Y_k$, with

$$Y_m = Y(\kappa_m + o(1))\mathcal{L}^{-1} \quad (1 \le m \le k) \tag{2.9}$$

where

$$\kappa_m \ge \kappa > 0 \quad (1 \le m \le k). \tag{2.10}$$

Suppose that $\rho(n) = 0$ unless $(n, P(N^{\theta/2})) = 1$, and

$$\sum_{q \le N^\theta} \mu^2(q) \tau_{3k}(q) \max_{(a,q)=1} \left| \sum_{n \equiv a \pmod q} \rho(n) X((\mathcal{A} + h_m) \cap \mathcal{A}; n) - \frac{Y_m}{\phi(q)} \right| \ll Y \mathcal{L}^{-k-\varepsilon} \tag{2.11}$$

for $1 \le m \le k$. Finally, suppose that

$$\log k - C_5 > \frac{2t-2}{\kappa\theta} + \varepsilon. \tag{2.12}$$

Then there is a set $\mathcal{S}$ in $\mathbb{P} \cap \mathcal{A}$ such that $\mathcal{S} - \mathcal{R}$ is square-free and

$$\# \mathcal{S} = t, \qquad D(\mathcal{S}) \leq h_k - h_1.$$

If $Y > N^{1/2+\varepsilon}$, the assertion of the theorem is also valid with (2.6) *replaced by*

$$1 \leq d \leq (MY^{-1})^2 N^{2\varepsilon}. \tag{2.13}$$

A few remarks may help here. Clearly $\mathcal{A}$ has got to possess many translations $\mathcal{A} + h$ such that $\mathcal{A} \cap (\mathcal{A} + h)$ contains, to within a constant factor, as many primes as $\mathcal{A}$. This rules out some sets $\mathcal{A}$ that we might wish to study, but does work in Theorems 2–4. The condition (2.11) is essentially a Bombieri-Vinogradov style theorem for primes in arithmetic progressions, and is usually much harder to establish for a given $\mathcal{A}$ than the requirement (2.7) on *integers* in arithmetic progressions.

For the proof of Theorem 5, which we now outline, we introduce 'Maynard weights' w_n $(n \in \mathbb{N})$. Let $R = N^{\theta/2-3}$ and $K = (s+1)k+1$. Let

$$W_1 = P^2 \prod_{K < p \leq D_0} p.$$

We define weights $y_{\boldsymbol{r}}$ and $\lambda_{\boldsymbol{r}}$ as follows for $\boldsymbol{r} = (r_1, \ldots, r_k) \in \mathbb{N}^k$: $y_{\boldsymbol{r}} = \lambda_{\boldsymbol{r}} = 0$ unless

$$\left(\prod_{i=1}^{k} r_i, W_1 \right) = 1 \ , \ \mu^2 \left(\prod_{i=1}^{k} r_i \right) = 1. \tag{2.14}$$

If (2.14) holds, let

$$y_{\boldsymbol{r}} = F\left(\frac{\log r_1}{\log R}, \ldots, \frac{\log r_k}{\log R} \right). \tag{2.15}$$

Now $\lambda_{\boldsymbol{d}}$ is defined by

$$\lambda_{\boldsymbol{d}} = \prod_{i=1}^{k} \mu(d_i) d_i \sum_{\substack{\boldsymbol{r} \\ d_i | r_i \ \forall i}} \frac{y_{\boldsymbol{r}}}{\prod\limits_{i=1}^{k} \phi(r_i)}. \tag{2.16}$$

We pick a suitable integer $\nu_0 = \nu_0(\mathcal{R}, \mathcal{H})$; see Section 3 for the details. For $n \equiv \nu_0 \pmod{W_1}$, let

$$w_n = \left(\sum_{d_i | n + h_i \ \forall i} \lambda_{\boldsymbol{d}} \right)^2.$$

For other $n \in \mathbb{N}$, let $w_n = 0$. Let

$$S_1 = \sum_{\substack{N \leq n < 2N \\ n \in \mathcal{A}}} w_n, \tag{2.17}$$

$$S_2(m) = \sum_{\substack{N \leq n < 2N \\ n \in \mathcal{A} \cap (\mathcal{A} - h_m)}} w_n \rho(n + h_m). \tag{2.18}$$

We shall obtain the asymptotic formulas

$$S_1 = \frac{(1+o(1))\phi(W_1)^k Y(\log R)^k I_k(F)}{W_1^{k+1}}, \tag{2.19}$$

$$S_2(m) = \frac{(1+o(1))\kappa_m\,\phi(W_1)^k Y(\log R)^{k+1} J_k^{(m)}(F)}{W_1^{k+1}\mathcal{L}} \tag{2.20}$$

as $N \to \infty$. To see how to make use of this, let us introduce a probability measure on $\mathcal{A}$ defined by

$$Pr\{n\} = \frac{w_n}{S_1} \quad (n \in \mathcal{A}). \tag{2.21}$$

It is not a very long step from (2.19), (2.20) to show that

$$Pr\left(\sum_{m=1}^{k} X(\mathbb{P}\cap\mathcal{A};\ n+h_m) \geq t\right) > \varepsilon/k. \tag{2.22}$$

We will now reach our goal by showing that

$$Pr(n+h_m-b_\ell \text{ is not squarefee}) \ll D_0^{-1} \tag{2.23}$$

for given $h_m \in \mathcal{H}_k$ and $b_\ell \in \mathcal{R}$. For then there is a probability greater than $\varepsilon/2k$ that at least t of $n+h_1,\dots,n+h_k$ are primes p in $\mathcal{A}$ for which $p-\mathcal{R}$ is squarefree.

To obtain (2.23), we give upper bounds for the quantities

$$\Omega(p) := \sum\{w_n : n\in\mathcal{A}, p^2 \mid n+h_m-b_\ell\} \qquad (p\in\mathbb{P}) \tag{2.24}$$

Our choice of ν_0 will show at once that

$$\Omega(p) = 0 \qquad (p \leq D_0). \tag{2.25}$$

Primes p in $(D_0, B]$, for a suitable B, are treated by an analysis similar to the discussion of S_1. Then we 'aggregate' primes $p > B$ by bounding

$$S_{m,\ell} := \sum_{\substack{n\,\in\mathcal{A}\\ p^2\mid n+h_m-b_\ell\ (\text{some } p>B)}} w_n \tag{2.26}$$

to reach (2.23).

We retain the notations introduced in this section in Section 3, where the above outline is filled out to a complete proof of Theorem 5.

3. Proof of Theorem 5

We first show that there is an integer ν_0 with

$$(\nu_0 + h_m, W_1) = 1 \quad (1 \le m \le k) \tag{3.1}$$

$$p^2 \nmid \nu_0 + h_m - b_\ell \quad (p \le K,\ 1 \le \ell \le s,\ 1 \le m \le k) \tag{3.2}$$

and

$$p \nmid \nu_0 + h_m - b_\ell \quad (K < p \le D_0,\ 1 \le \ell \le s,\ 1 \le m \le k). \tag{3.3}$$

By the Chinese remainder theorem, it suffices to specify $\nu_0 \pmod{p^2}$ for $p \le K$ and $\nu_0 \,(\operatorname{mod} p)$ for $K < p \le D_0$. We use $h_j \equiv 0 \pmod{p^2}$ $(p \le K)$. The property (3.1) reduces to

$$\nu_0 \not\equiv 0 \pmod{p} \quad (p \le K) \tag{3.4}$$

and

$$\nu_0 + h_m \not\equiv 0 \pmod{p} \quad (K < p \le D_0,\ 1 \le m \le k). \tag{3.5}$$

We define $b_0 = 0$. Now (3.2), (3.3), (3.4), (3.5) can be rewritten as

$$\nu_0 \not\equiv 0 \pmod{p},\ \nu_0 \not\equiv b_\ell \pmod{p^2} \ \ (p \le K, 1 \le \ell \le s), \tag{3.6}$$

$$\nu_0 + h_m - b_\ell \not\equiv 0 \,(\operatorname{mod} p)\ (K < p \le D_0, 0 \le \ell \le s, 1 \le m \le k). \tag{3.7}$$

For (3.6), we select ν_0 in a reduced residue class $(\operatorname{mod} p^2)$ not occupied by b_ℓ $(1 \le \ell \le s)$. For (3.7), we observe that ν_0 can be chosen from the $p-1$ reduced residue classes $(\operatorname{mod} p)$, avoiding at most $(s+1)k$ classes, since $p - 1 > (s+1)k$.

To save space, we refer to arguments in [3, 13, 14] in our proof.

Exactly as in the proof of [3, Proposition 1] with $q_0 = 1$, $W_2 = W_1$, we find that the asymptotic formulas (2.19), (2.20) hold as $N \to \infty$. (The value of W_1 in [3] is $\prod\limits_{p \le D_0} p$, but this does not affect the proof.)

Exactly as in [3] following the statement of Proposition 2, we derive from (2.19), (2.20), (2.8), (2.4), (2.12), the inequality

$$\sum_{m=1}^{k} \sum_{n \in \mathcal{A}} w_n X(\mathbb{P} \cap \mathcal{A},\, n + h_m) > (t - 1 + \varepsilon) \sum_{n \in \mathcal{A}} w_n. \tag{3.8}$$

Writing $\mathbb{E}[\cdot]$ for expectation for the probability measure $Pr\{n\}$, (3.8) becomes

$$\mathbb{E}\left[\sum_{m=1}^{k} X(\mathbb{P} \cap \mathcal{A};\ n + h_m)\right] > t - 1 + \varepsilon.$$

It is easy to deduce that

$$Pr\left(\sum_{m=1}^{k} X(\mathbb{P} \cap \mathcal{A};\ n + h_m) \ge t\right) > \frac{\varepsilon}{k}.$$

As explained above, it remains to prove (2.23) for a given pair m, ℓ.

The upper bound

$$\sum_{\substack{N \le n < N+M \\ n \equiv \nu_0 \pmod{W_1}}} w_n^2 \ll \mathcal{L}^{19k} \frac{M}{W_1} + N^{2\theta} \tag{3.9}$$

can be proved in exactly the same way as [13, (3.10)].

Let

$$B = (MY^{-1})^2 \max(\mathcal{L}^{18k}, N^{2\theta} M^{-1}).$$

Clearly

$$\Pr(n + h_m - b_\ell \text{ is not square-free}) \le \frac{1}{S_1}\left(\sum_{p \le B} \Omega(p) + S_{m,\ell}\right).$$

To obtain (2.23) we need only show that

$$\sum_{p \le B} \Omega(p) \ll \frac{\phi(W_1)^k Y \mathcal{L}^k}{W_1^{k+1} D_0} \tag{3.10}$$

and

$$S_{m,\ell} \ll \frac{\phi(W_1)^k Y \mathcal{L}^k}{W_1^{k+1} D_0} \tag{3.11}$$

From (3.1)–(3.3), $\Omega(p) = 0$ for $p \le D_0$. Take $D_0 < p \le B$. We have

$$\Omega(p) = \sum_{\boldsymbol{d},\boldsymbol{e}} \lambda_{\boldsymbol{d}} \lambda_{\boldsymbol{e}} \sum_{\substack{n \in \mathcal{A} \\ n \equiv \nu_0 \pmod{W_1} \\ n \equiv b_\ell - h_m \pmod{p^2} \\ n \equiv -h_i \pmod{[d_i, e_i]}\ \forall i}} 1. \tag{3.12}$$

Fix $\boldsymbol{d}$, $\boldsymbol{e}$ with $\lambda_{\boldsymbol{d}} \lambda_{\boldsymbol{e}} \neq 0$. The inner sum in (3.12) is empty if $(d_i, e_j) > 1$ for a pair i, j with $i \neq j$ (compare [3, §2]). The inner sum is also empty if $p \mid [d_i, e_i]$ since then

$$p \mid n + h_i - (n + h_m - b_\ell) = h_m - h_i - b_\ell$$

which is absurd, since $h_m - h_i - b_\ell$ is bounded and is nonzero by hypothesis.

We may now replace (3.12) by

$$\Omega(p) = \sideset{}{'}\sum_{\substack{\boldsymbol{d},\boldsymbol{e} \\ (d_i,p)=(e_i,p)=1\ \forall i}} \lambda_{\boldsymbol{d}} \lambda_{\boldsymbol{e}} \left\{ \frac{Y}{p^2 W_1 \prod_{i=1}^k [d_i, e_i]} + O\left(V\left(p^2 W_1 \prod_{i=1}^k [d_i, e_i]\right)\right)\right\}, \tag{3.13}$$

where $\sum'$ denotes a summation restricted by: $(d_i, e_j) = 1$ whenever $i \neq j$. Expanding the right-hand side of (3.13), we obtain a main term of the shape estimated in

Lemma 2.5 of [14]. The argument there gives

$$\sideset{}{'}\sum_{\substack{\boldsymbol{d},\boldsymbol{e}\\(d_i,p)=(e_i,p)=1\,\forall i}} \frac{\lambda_{\boldsymbol{d}}\lambda_{\boldsymbol{e}}}{\prod_{i=1}^k [d_i,e_i]} = \sideset{}{'}\sum_{\boldsymbol{d},\boldsymbol{e}} \frac{\lambda_{\boldsymbol{d}}\lambda_{\boldsymbol{e}}}{\prod_{i=1}^k [d_i,e_i]} + O\left(\frac{1}{p}\left(\frac{\phi(W)}{W}\mathcal{L}\right)^k\right),$$

uniformly for $p > D_0$. As already alluded to above in the discussion of S_1, the behavior of the main term here can be read out of the proof of [3, Proposition 1]. Collecting our estimates, we find that

$$\sideset{}{'}\sum_{\substack{\boldsymbol{d},\boldsymbol{e}\\(d_i,p)=(e_i,p)=1\,\forall i}} \frac{\lambda_{\boldsymbol{d}}\lambda_{\boldsymbol{e}}}{\prod_{i=1}^k [d_i,e_i]} = \frac{\phi(W_1)^k}{W_1^k}(\log R)^k I_k(F)(1+o(1)).$$

Clearly this gives

$$\sum_{D_0<p\le B}\Omega(p) \ll \frac{Y\phi(W_1)^k}{W_1^{k+1}}\mathcal{L}^k \sum_{p>D_0} p^{-2} + (\max_{\boldsymbol{d}}|\lambda_{\boldsymbol{d}}|)^2 \sum_{D_0<p\le B}\ \sum_{\ell\le R^2W_1} \mu^2(\ell)\tau_{3k}(\ell)V(p^2\ell).$$

(We use (3.13) along with a bound for the number of occurrences of ℓ as $W_1\prod_{i=1}^k [d_i,e_i]$.) It is not difficult to see that $\lambda_{\boldsymbol{d}} \ll \mathcal{L}^k$ (compare [11], (5.9)). On an application of (2.7) with $d = p^2$ satisfying (2.6), we obtain the bound (3.10).

Let $\sum_{n;\,(3.14)}$ denote a summation over n with

$$N \le n < N+M,\ n \equiv \nu_0 \pmod{W_1},\ p^2 \mid n + h_m - b_\ell \text{ (some } p > B). \tag{3.14}$$

Cauchy's inequality gives

$$\begin{aligned} S_{m,\ell} &\le \sum_{n;\,(3.14)} w_n \\ &\le \left(\sum_{n;\,(3.14)} 1\right)^{1/2} \left(\sum_{\substack{n\equiv\nu_0 \,(\mathrm{mod}\, W_1)\\ N\le n<N+M}} w_n^2\right)^{1/2} \\ &\ll \left(\sum_{B<p\le(3N)^{1/2}} \left(\frac{M}{p^2W_1}+1\right)\right)^{1/2} \left(\frac{M^{1/2}}{W_1^{1/2}}\mathcal{L}^{19k/2} + N^\theta\right) \end{aligned}$$

(by (3.9))

$$\ll \frac{M\mathcal{L}^{19k/2}}{W_1 B^{1/2}} + \frac{N^\theta M^{1/2}}{W_1^{1/2}B^{1/2}} + \frac{M^{1/2}N^{1/4}\mathcal{L}^{19k/2}}{W_1^{1/2}} + N^{\frac14+\theta}.$$

To complete the proof we verify (disregarding W_1) that each of these four terms is $\ll Y\mathcal{L}^{k-1/2}$. We have

$$M\mathcal{L}^{19k/2}B^{-1/2}(Y\mathcal{L}^{k-1/2})^{-1} \ll 1$$

since $B \geq \mathcal{L}^{18k}(MY^{-1})^2$. We have

$$N^{\theta}M^{1/2}B^{-1/2}(Y\mathcal{L}^{k-1/2})^{-1} \ll 1$$

since $B \geq (MY^{-1})^2\ N^{2\theta}M^{-1}$. We have

$$M^{1/2}N^{1/4}\mathcal{L}^{19k/2}(Y\mathcal{L}^{k-1/2})^{-1} \ll 1$$

since $Y \gg N^{1/4}\mathcal{L}^{9k}M^{1/2}$. Finally,

$$N^{1/4+\theta}(Y\mathcal{L}^{k-1/2})^{-1} \ll 1$$

since $Y \gg N^{\theta+1/4}$. This completes the proof of the first assertion of Theorem 5.

Now suppose $Y > N^{\frac{1}{2}+\varepsilon}$. We can replace B by $B_1 := (MY^{-1})N^{\varepsilon}$ throughout, and at the last stage of the proof use the bound

$$S_{m,\ell} \leq w \sum_{\substack{N \leq n \leq N+M \\ p^2 | n+h_m-b_\ell \\ (\text{some } p > B_1)}} 1, \tag{3.15}$$

where

$$w := \max_n w_n.$$

Now

$$w = \sum_{[d_i,e_i] | n_1+h_i\ \forall i} \lambda_{\boldsymbol{d}}\lambda_{\boldsymbol{e}}$$

for some choice of $n_1 \leq N+M$. The number of possibilities for $d_1, e_1, \ldots, d_k, e_k$ in this sum is $\ll N^{\varepsilon/3}$. Hence (3.15) yields

$$\begin{aligned} S_{m,\ell} &\ll N^{\varepsilon/2} \sum_{B_1 < p \leq 3N^{1/2}} \left(\frac{M}{p^2} + 1\right) \\ &\ll \frac{N^{\varepsilon/2}M}{B_1} + N^{1/2+\varepsilon/2} \ll Y\mathcal{L}^{k-1/2}. \end{aligned}$$

The second assertion of Theorem 5 follows from this. $\square$

4. Proof of Theorems 2 and 3.

We begin with Theorem 2, taking $\kappa = \kappa_m = 1$, $\rho(n) = X(\mathbb{P}; n)$, $M = Y = N^{\phi}$, $Y_m = \int_N^{N+M} \frac{dt}{\log t}$. By results of Timofeev [19], we find that (2.11) holds with $\theta = \psi$. Since $2\psi < \phi$, the range of d given by (2.6) is

$$d \ll \mathcal{L}^{36k}. \tag{4.1}$$

Now (2.7) is a consequence of the elementary bound $V(m) \ll 1$.

Turning to the construction of a compatible set $\mathcal{H}_k$, let $L = 2(k-1)s + 1$. Take the first L primes $q_1 < \cdots < q_L$ greater than L. Select $q_1' = q_1, q_2', \ldots, q_k'$ recursively from $\{q_1, \ldots, q_L\}$ so that q_j satisfies

$$P^2 q_j' \neq P^2 q_i' \pm b_\ell \quad (1 \le i \le j-1,\ 1 \le \ell \le s), \tag{4.2}$$

a choice which is possible since $L > 2(j-1)s$. Now $\mathcal{H}_k = \{P^2 q_1', \ldots, P^2 q_k'\}$ is an admissible set compatible with $\mathcal{R}$. The set $\mathcal{S}$ given by Theorem 5 satisfies

$$D(\mathcal{S}) \le P^2(q_L - q_1) \ll \exp(O(ks)).$$

As for the choice of k, the condition (2.12) is satisfied when

$$k = \left\lceil \exp\left(\frac{2t}{\psi} + C_5\right)\right\rceil + 1.$$

Theorem 2 follows at once.

For Theorem 3, we adapt the proof of [3, Theorem 3]. Let $\gamma = \alpha^{-1}$, $N = M = v^2$ and $\theta = \frac{2}{7} - \varepsilon$. We take

$$\mathcal{A} = \{n \in [N, 2N) : n = \lfloor \alpha m + \beta \rfloor \text{ for some } m \in \mathbb{N}\} \quad \text{and} \quad Y = \gamma N.$$

We find as in [3] that

$$\mathcal{A} = \{n \in [N, 2N) : \gamma n \in I \pmod 1\},$$

where $I = (\gamma\beta - \gamma, \gamma\beta]$. The properties that we shall enforce in constructing $h_1, \ldots, h_k$ are

(i) $h_1, \ldots, h_k$ is compatible with $\mathcal{R}$;

(ii) we have $h_m = h_m' + h$ $(1 \le m \le k)$, where $h\gamma \in (\eta - \varepsilon\gamma, \eta) \pmod 1$ and

$$-\gamma h_m' \in (\eta, \eta + \varepsilon\gamma) \pmod 1 \quad \text{for some real } \eta;$$

(iii) we have

$$\log k - C_5 > \frac{2t - 2}{0.90411\left(\frac{2}{7} - \varepsilon\right)}.$$

The condition (ii) gives us enough information to establish (2.11); here we follow [3] verbatim, using the function $\rho = \rho_1 + \rho_2 + \rho_3 - \rho_4 - \rho_5$ in [3, Lemma 18], and taking κ slightly larger than 0.90411 in (2.10).

Turning to (2.7), with the range of d as in (4.1), we may deduce this bound from [3, Lemma 12] with $M = d$, $a_m = 1$ for $m = d$, $a_m = 0$ otherwise, $Q \le N^{2/7-\varepsilon}$, $K = N/d$ and $H = \mathcal{L}^{A+1}$. This requires an examination of the reduction to mixed sums in [3, Section 5].

It remains to obtain $h_1, \dots, h_k$ satisfying (i)–(iii) above. We use the following lemma.

Lemma 1. *Let I be an interval of length ν, $0 < \nu < 1$. Let $x_1, \dots, x_J$ be real and $a_1, \dots, a_J$ positive.*

(a) There exists z such that

$$\#\{j \le J : x_j \in z + I \pmod 1\} \ge J\nu.$$

(b) For any $L \in \mathbb{N}$, we have

$$\Bigg| \sum_{\substack{j=1 \\ x_j \in I \ (\mathrm{mod}\, 1)}}^{J} a_j - \nu \cdot \sum_{j=1}^{J} a_j \Bigg| \le \frac{1}{L+1} \sum_{j=1}^{J} a_j + 2 \sum_{m=1}^{L} \left(\frac{1}{L+1} + \nu \right) \Bigg| \sum_{j=1}^{J} a_j e(m x_j) \Bigg|.$$

Proof. We leave (a) as an exercise. Let $T_1(\theta) = \sum_{m=-L}^{L} \widehat{T}_1(m) e(m\theta)$ be the trigonometric polynomial in [1, Lemma 2.7]. We obtain (b) by a simple modification of the proof of [1], Theorem 2.1 on revising the upper bound for $|\widehat{T}_1(m)|$:

$$|\widehat{T}_1(m)| \le \frac{1}{L+1} + \frac{|\sin \pi \nu m|}{\pi m} \le \frac{1}{L+1} + \nu. \qquad \square$$

Now let ℓ be the least integer with

$$\log(\varepsilon \gamma \ell) \ge \frac{2t-2}{0.90411\left(\frac{2}{7} - \varepsilon\right)} + C_5, \tag{4.3}$$

and let $L = 2(\ell-1)s+1$. As above, select primes $q_1', \dots, q_\ell'$ from $q_1, \dots, q_L$ so that (4.2) holds. Applying Lemma 1, choose $h_1', \dots, h_k'$ from $\{P^2 q_1', \dots, P^2 q_\ell'\}$ so that, for some real η,

$$-\gamma h_m' \in (\eta, \eta + \varepsilon\gamma) \pmod 1 \quad (m = 1, \dots, k)$$

and

$$k \ge \varepsilon \gamma \ell. \tag{4.4}$$

We combine (4.3), (4.4) with (2.12) to obtain (iii). Now there is a bounded h, $h \equiv 0 \pmod{P^2}$, with

$$\gamma h \in (\eta - \varepsilon\gamma, \eta) \pmod 1.$$

This follows from Lemma 1 with $x_j = jP^2\gamma$, since

$$\sum_{j=1}^{J} e(mjP^2\gamma) \ll \frac{1}{\|mP^2\gamma\|}.$$

We now have (i), (ii) and (iii). Theorem 5 yields the required set of primes $\mathcal{S}$ with

$$D(\mathcal{S}) \le P^2(q_L - q_1) \ll \exp(O(\ell s)),$$

and the desired bound (1.10) follows from the choice of ℓ. This completes the proof of Theorem 3.

5. Lemmas for the proof of Theorem 4

We begin by extending a theorem of Robert and Sargos [18] (essentially, their result is the case $Q = 1$ of Lemma 2).

Lemma 2. *Let $H \ge 1$, $N \ge 1$, $M \ge 1$, $Q \ge 1$, $X \gg HN$. For $H < h \le 2H$, $N < n \le 2N$, $M < m \le 2M$ and the characters $\chi \pmod q$, $1 \le q \le Q$, let $a(h, n, q, \chi)$ and $g(m)$ be complex numbers,*

$$|a(h, n, q, \chi)| \le 1, \quad |g(m)| \le 1.$$

Let α, β, γ be fixed real numbers, $\alpha(\alpha - 1)\beta\gamma \ne 0$. Let

$$S_0(\chi) = \sum_{H<h\le 2H} \sum_{N<n\le 2N} a(h, n, q, \chi) \sum_{M<m\le 2M} g(m)\chi(m)e\left(\frac{Xh^\beta n^\gamma m^\alpha}{H^\beta N^\gamma M^\alpha}\right).$$

Then

$$\sum_{q\le Q} \sum_{\chi \,(\mathrm{mod}\, q)} |S_0(\chi)| \ll (HMN)^\varepsilon \left(Q^2 HNM^{\frac{1}{2}} + Q^{3/2} HNM \left(\frac{X^{\frac{1}{4}}}{(HN)^{\frac{1}{4}} M^{\frac{1}{2}}} + \frac{1}{(HN)^{\frac{1}{4}}} \right) \right).$$

Proof. By Cauchy's inequality,

$$|S_0(\chi)|^2 \le HN \sum_{H<h\le 2H} \sum_{N<n\le 2N} \sum_{\substack{M<m_1\le 2M \\ M<m_2\le 2M}} g(m_1)\overline{g(m_2)}\chi(m_1)\overline{\chi(m_2)}e(Xu(h, n)v(m_1, m_2)),$$

with

$$u(h, n) = \frac{h^\beta n^\gamma}{H^\beta N^\gamma}, \quad v(m_1, m_2) = \frac{m_1^\alpha - m_2^\alpha}{M^\alpha}.$$

Summing over χ,

$$\sum_{\chi \,(\mathrm{mod}\, q)} |S_0(\chi)|^2 \le HN \sum_{H<h\le 2H} \sum_{N<n\le 2N} \phi(q) \sum_{\substack{M<m_1\le 2M\\ M<m_2\le 2M\\ m_1\equiv m_2 \,(\mathrm{mod}\, q)}} g(m_1)\overline{g(m_2)} e(Xu(h,n)v(m_1,m_2)).$$

Separating the contribution from $m_1 = m_2$, and summing over q,

$$\sum_{q\le Q} \sum_{\chi \,(\mathrm{mod}\, q)} |S_0(\chi)|^2 \le H^2N^2M \sum_{q\le Q} \phi(q) + S_1,$$

where

$$S_1 = C(\varepsilon)M^\varepsilon QHN \sum_{H<h\le 2H} \sum_{N<n\le 2N} \sum_{\substack{M<m_1\le 2M\\ M<m_2\le 2M}} w(m_1,m_2) e(Xu(h,n)v(m_1,m_2)),$$

with

$$w(m_1,m_2) = \begin{cases} 0 & \text{if } m_1 = m_2,\\ \displaystyle\sum_{q\le Q} \sum_{m_1-m_2=qn,\, n\in\mathbb{Z}} \frac{g(m_1)\overline{g(m_2)}\phi(q)}{C(\varepsilon)M^\varepsilon Q} & \text{if } m_1 \ne m_2. \end{cases}$$

Note that

$$|w(m_1,m_2)| \le 1$$

for all m_1, m_2 if $C(\varepsilon)$ is suitably chosen.

We now apply the double large sieve to S_1 exactly as in [18, (6.5)]. Using the upper bounds given in [18], we have

$$S_1 \ll M^\varepsilon QHNX^{1/2}\mathcal{B}_1^{1/2}\mathcal{B}_2^{1/2},$$

where

$$\mathcal{B}_1 = \sum_{\substack{h_1,n_1,h_2,n_2\\ |u(h_1,n_1)-u(h_2,n_2)|\le 1/X\\ H<h_i\le 2H,\, N<n_i\le 2N \ (i=1,2)}} 1 \ll (HN)^{2+\varepsilon}\left(\frac{1}{HN}+\frac{1}{X}\right) \ll (HN)^{1+\varepsilon},$$

and

$$\mathcal{B}_2 = \sum_{\substack{m_1,m_2,m_3,m_4\\ |v(m_1,m_2)-v(m_3,m_4)|\le 1/X\\ M<m_i\le 2M \ (1\le i\le 4)}} 1 \ll M^{4+\varepsilon}\left(\frac{1}{M^2}+\frac{1}{X}\right).$$

Hence

$$\sum_{q\le Q}\ \sum_{\chi \,(\bmod q)} |S_0(\chi)|^2 \ll Q^2H^2N^2M+(MHN)^{2+2\varepsilon}Q\left(\frac{X^{1/2}}{(HNM^2)^{1/2}}+\frac{1}{(HN)^{1/2}}\right).$$

Lemma 2 follows on an application of Cauchy's inequality. □

Lemma 3. *Fix c, $0<c<1$. Let $h\ge 1$, $m\ge 1$, $K>1$, $K'\le 2K$,*

$$S=\sum_{K<k\le K',\, mk\equiv u \pmod q} e(h(mk)^c).$$

Then for any q, u,

$$S \ll (hm^cK^c)^{1/2}+K(hm^cK^c)^{-1/2}.$$

Proof. We write S in the form

$$\begin{aligned} S &= \frac{1}{q}\sum_{K<k\le K'}\sum_{r=1}^{q} e\left(\frac{r(mk-u)}{q}+h(mk)^c\right)\\ &= \frac{1}{q}\sum_{r=1}^{q} e\left(-\frac{ur}{q}\right)\sum_{K<k\le K'} e\left(\frac{rmk}{q}+h(mk)^c\right), \end{aligned}$$

and apply [9, Theorem 2.2] to each sum over k. □

6. Proof of Theorem 4

Throughout this section, fix $c\in\left(\frac{8}{9},1\right)$ and define, for an interval I of length $|I|<1$,

$$\mathcal{A}(I)=\{n\in[N,2N): n^c\in I \pmod 1\}.$$

We choose $\mathcal{H}_k$ compatible with $\mathcal{R}$ as in the proof of Theorem 2, so that

$$h_k-h_1 \ll \exp(O(ks)).$$

We apply the second assertion of Theorem 5 with

$$M=N,\quad Y=N^{c+\varepsilon},\quad \kappa=1,\quad \rho(n)=X(\mathbb{P};n).$$

We define θ by

$$\theta=\frac{9c-8}{6}-\varepsilon,$$

and we choose $k=\lceil \exp(\frac{2t-2}{\theta}+C_5)\rceil+1$, so that (2.12) holds. By our choice of θ, the range in (2.13) is contained in

$$(6.1)\qquad\qquad 1\le d\le N^{2-2c}.$$

It remains to verify (2.7) and (2.11) for a fixed h_m. We consider (2.11) first.

The set $(\mathcal{A}+h_m)\cap\mathcal{A}$ consists of those n in $[N,2N)$ with

$$n^c-\beta\in[0,N^{-1+c+\varepsilon})\ (\mathrm{mod}\,1),\ \ (n+h_m)^c-\beta\in[0,N^{-1+c+\varepsilon})\ (\mathrm{mod}\,1).$$

Since

$$(n+h_m)^c=n^c+O(N^{c-1})\quad(N\le n<2N),$$

we have

$$\mathcal{A}(I_2)\subset(\mathcal{A}+h_m)\cap\mathcal{A}\subset\mathcal{A}(I_1) \tag{6.2}$$

where, for a given A,

$$\begin{aligned}I_1&=[\beta,\beta+N^{-1+c+\varepsilon}),\\ I_2&=[\beta,\beta+N^{-1+c+\varepsilon}\ (1-\mathcal{L}^{-A-3k})).\end{aligned}$$

By a standard partial summation argument it will suffice to show that, for any choice of u_q relatively prime to q,

$$\sum_{q\le N^\theta}\mu^2(q)\tau_{3k}(q)\Bigg|\sum_{\substack{n\,\equiv\,u_q\ (\mathrm{mod}\,q)\\ N\le n<N'}}\left(\Lambda(n)X((\mathcal{A}+h_m)\cap\mathcal{A};n)-N^{-1+c+\varepsilon}\frac{q}{\phi(q)}\right)\Bigg|\ll Y\mathcal{L}^{-A}$$

for $N'\in[N,2N)$. (The implied constant here and below may depend on A.) In view of (6.2), we need only show that for any $A>0$,

$$\sum_{q\le N^\theta}\mu^2(q)\tau_{3k}(q)\Bigg|\sum_{\substack{n\,\equiv\,u_q\ (\mathrm{mod}\,q)\\ N\le n<N'}}\left(\Lambda(n)X(\mathcal{A}(I_j);n)-N^{-1+c+\varepsilon}\frac{q}{\phi(q)}\right)\Bigg|\ll Y\mathcal{L}^{-A}\ (j=1,2). \tag{6.3}$$

The sum in (6.3) is bounded by $\sum_1+\sum_2$, where

$$\sum\nolimits_1=\sum_{q\le N^\theta}\mu^2(q)\tau_{3k}(q)\Bigg|\sum_{\substack{n\,\equiv\,u_q\ (\mathrm{mod}\,q)\\ n^c\in I_j\ (\mathrm{mod}\,1)\\ N\le n<N'}}\Lambda(n)-N^{-1+c+\varepsilon}\sum_{\substack{n\,\equiv\,u_q\ (\mathrm{mod}\,q)\\ N\le n<N'}}\Lambda(n)\Bigg|$$

and

$$\sum\nolimits_2=N^{-1+c+\varepsilon}\sum_{q\le N^\theta}\mu^2(q)\tau_{3k}(q)\Bigg|\sum_{\substack{n\,\equiv\,u_q\ (\mathrm{mod}\,q)\\ N\le n<N'}}\left(\Lambda(n)-\frac{q}{\phi(q)}\right)\Bigg|.$$

Deploying the Cauchy-Schwarz inequality in the same way as in [11, (5.20)], it follows from the Bombieri-Vinogradov theorem that

$$\sum\nolimits_2\ll N^{c+\varepsilon}\mathcal{L}^{-A}.$$

Moreover,

$$\sum_{q\le N^\theta}\mu^2(q)\tau_{3k}(q)\left|N^{-1+c+\varepsilon}\sum_{\substack{n\,\equiv\,u_q \pmod q\\ N\le n<N'}}\Lambda(n)-|I_j|\sum_{\substack{n\,\equiv\,u_q \pmod q\\ N\le n<N'}}\Lambda(n)\right|\ll N^{c+\varepsilon}\mathcal{L}^{-A}$$

(trivially for $j=1$, and by the Brun-Titchmarsh inequality for $j=2$). Thus it remains to show that

$$\sum_{q\le N^\theta}\mu^2(q)\tau_{3k}(q)\left|\sum_{\substack{n\,\equiv\,u_q \pmod q\\ n^c\in I_j \pmod 1\\ N\le n<N'}}\Lambda(n)-|I_j|\sum_{\substack{n\,\equiv\,u_q \pmod q\\ N\le n<N'}}\Lambda(n)\right|\ll N^{c+\varepsilon}\mathcal{L}^{-A}.$$

Let $H=N^{1-c-\varepsilon}\mathcal{L}^{A+3k}$. We apply Lemma 1, with $a_j=\Lambda(N+j-1)$ for $N+j-1\equiv u_q \pmod q$ and $a_j=0$ otherwise, and $L=H$. Using the Brun-Titchmarsh inequality, we find that

$$\left|\sum_{\substack{n\,\equiv\,u_q \pmod q\\ n^c\in I_j \pmod 1\\ N\le n<N'}}\Lambda(n)-|I_j|\sum_{\substack{n\,\equiv\,u_q \pmod q\\ N\le n<N'}}\Lambda(n)\right|$$
$$\ll\frac{N^{c+\varepsilon}}{\phi(q)}\mathcal{L}^{-A-3k}+N^{-1+c+\varepsilon}\sum_{1\le h\le H}\left|\sum_{\substack{N\le n<N'\\ n\,\equiv\,u_q \,(\mathrm{mod}\, q)}}\Lambda(n)e(hn^c)\right|.$$

Recalling the upper estimate $\tau_{3k}(q)\ll N^{\varepsilon/20}$ for $q\le N^\theta$, it suffices to show that

$$\sum_{q\le N^\theta}\sum_{1\le h\le H}\sigma_{q,h}\sum_{\substack{N\le n<N'\\ n\,\equiv\,u_q \,(\mathrm{mod}\, q)}}\Lambda(n)e(hn^c)\ll N^{1-\varepsilon/10}$$

for complex numbers $\sigma_{q,h}$ with $|\sigma_{q,h}|\le 1$.

We apply a standard dyadic dissection argument, finding that it suffices to show that

$$\sum_{q\le N^\theta}\sum_{H_1\le h\le 2H_1}\sigma_{q,h}\sum_{\substack{N\le n<N'\\ n\,\equiv\,u_q \,(\mathrm{mod}\, q)}}\Lambda(n)e(hn^c)\ll N^{1-\varepsilon/9} \tag{6.4}$$

for $1\le H_1\le H$. The next step is a standard decomposition of the von Mangoldt function; see for example [6, Section 24]. In order to obtain (6.4), it suffices to show, under each of two sets of conditions on M, K, $(g_k)_{k\in[K,2K)}$, that

$$\sum_{q\le N^\theta}\sum_{H_1\le h\le 2H_1}\sigma_{q,h}\sum_{\substack{M\le m<2M\\ N\le mk<N'\\ mk\,\equiv\,u_q \,(\mathrm{mod}\, q)}}\sum_{K\le k<2K}a_m g_k e(h(mk)^c)\ll N^{1-\varepsilon/8} \tag{6.5}$$

for complex numbers a_m, g_k with $|a_m| \le 1, |g_k| \le 1$. The first set of conditions is

$$N^{1/2} \ll K \ll N^{2/3}. \tag{6.6}$$

The second set of conditions is

$$K \gg N^{2/3}, \quad g_k = \begin{cases} 1 & \text{if } K \le k < K', \\ 0 & \text{if } K' \le k < 2K. \end{cases} \tag{6.7}$$

We first obtain (6.5) under the condition (6.6). We replace (6.5) by

$$\sum_{q \le N^\theta} \frac{1}{\phi(q)} \sum_{\chi\,(\mathrm{mod}\,q)} \overline{\chi}(u_q) \sum_{H_1 \le h_1 \le 2H_1} \sigma_{q,h} \sum_{\substack{M \le m < 2M \\ N \le mk < N'}} \sum_{K \le k < 2K} a_m g_k \chi(m)\chi(k) e(h(mk)^c) \ll N^{1-\varepsilon/8}.$$

A further dyadic dissection argument reduces our task to showing that

(6.8)
$$\sum_{Q \le q \le 2Q} \sum_{\chi\,(\mathrm{mod}\,q)} \left| \sum_{H_1 \le h \le 2H_1} \sigma_{q,h} \sum_{M \le m < 2M} \sum_{K \le k < 2K} a_m g_k \chi(m)\chi(k) e(h(mk)^c) \right| \ll QN^{1-\varepsilon/7}$$

for $Q < N^\theta$.

We now apply Lemma 2 with $X = H_1 N^c$ and (H_1, K, M) in place of (H, N, M). The condition $X \gg H_1 K$ follows easily since $K \ll N^c$. Thus the left-hand side of (6.8) is

$$\begin{aligned} &\ll (H_1 N)^{\varepsilon/8} (Q^2 H_1 N^{1/2} K^{1/2} + Q^{3/2} H_1 N^{\frac{1}{2}+\frac{c}{4}} K^{1/4} + Q^{3/2} H_1^{3/4} N K^{-1/4}) \\ &\ll N^{\varepsilon/7} (Q^2 H_1 N^{5/6} + Q^{3/2} H_1 N^{2/3+c/4} + Q^{3/2} H_1^{3/4} N^{7/8}) \end{aligned}$$

using (6.6). Each term in the last expression is $\ll QN^{1-\varepsilon/7}$:

$$\begin{aligned} N^{\varepsilon/7} Q^2 H_1 N^{5/6} (QN^{1-\varepsilon/7})^{-1} &\ll N^{\theta+5/6-c+2\varepsilon/7} \ll 1, \\ N^{\varepsilon/7} Q^{3/2} H_1 N^{2/3+c/4} (QN^{1-\varepsilon/7})^{-1} &\ll N^{\theta/2+2/3-3c/4+2\varepsilon/7} \ll 1, \\ N^{\varepsilon/7} Q^{3/2} H_1^{3/4} N^{7/8} (QN^{1-\varepsilon/7})^{-1} &\ll N^{\theta/2+5/8-3c/4+2\varepsilon/7} \ll 1. \end{aligned}$$

We now obtain (6.5) under the condition (6.7). By Lemma 3, the left-hand side of (6.5) is

$$\begin{aligned} &\ll N^\theta M H_1 ((H_1 N^c)^{1/2} + K(H_1 N^c)^{-1/2}) \\ &\ll H_1^{3/2} N^{1+c/2+\theta} K^{-1} + H_1^{1/2} N^{1-c/2+\theta} \\ &\ll N^{11/6-c+\theta} + N^{3/2-c+\theta} \ll N^{1-\varepsilon/8}. \end{aligned}$$

Turning to (2.7), (under the condition (2.13) on d) by a similar argument to that leading to (6.5), it suffices to show that

$$\sum_{\substack{q \le N^\theta \\ (q,d)=1}} \sum_{H_1 \le h \le 2H_1} \left| \sum_{\substack{N \le n \le N' \\ n \equiv u_{qd} \,(\mathrm{mod}\, qd)}} e(hn^c) \right| \ll N^{1-\varepsilon/3} d^{-1} \tag{6.9}$$

for $d \le N^{2-2c}$, $H_1 \le N^{1-c}$, $N \le N' \le 2N$. By Lemma 3, the left-hand side of (6.9) is

$$\ll N^\theta H_1((H_1 N^c)^{1/2} + N(H_1 N^c)^{-1/2}).$$

Each of the two terms here is $\ll N^{1-\varepsilon/3} d^{-1}$. To see this,

$$N^\theta H_1^{3/2} N^{c/2} (N^{1-\varepsilon/3} d^{-1})^{-1} \ll N^{\theta+1/2-c} N^{2-2c} \ll 1$$

and

$$N^\theta H_1^{1/2} N^{1-c/2} (N^{1-\varepsilon/3} d^{-1})^{-1} \ll N^{\theta+1/2-c} N^{2-2c} \ll 1.$$

This completes the proof of Theorem 4. □

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