

A finiteness theorem for odd perfect numbers



Paul Pollack

University of Georgia

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Three kinds of natural numbers

Among simple even numbers, some are superabundant, others are deficient: these two classes are as two extremes opposed one to the other; as for those that occupy the middle point between the two, they are said to be perfect.

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Perfect: $s(n) = n$.

Ex: 5 is deficient ($s(5) = 1$), 12 is abundant ($s(12) = 16$), and 6 is perfect ($s(6) = 6$).

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. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.

Because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. – Donald Rumsfeld



Theorem (Euclid – Euler)

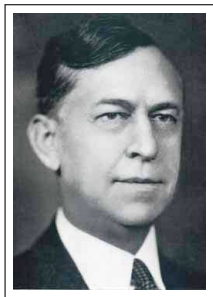
If $2^n - 1$ is prime, then

$$N := 2^{n-1}(2^n - 1)$$

is perfect. Conversely, if N is an even perfect number, then N has this form.

But what about odd perfect numbers?

Is there a simple formula for odd perfect numbers, like for even perfect numbers? Probably not.



Theorem (Dickson, 1913)

For each positive integer k , there are only finitely many odd perfect numbers with $\leq k$ distinct prime factors.

How do we understand $s(n)$ anyway?

It is more convenient to work with the sum of *all* positive divisors of n , including n itself, which is denoted $\sigma(n)$. That is,

$$\sigma(n) = \sum_{d|n} d.$$

Then $s(n) = \sigma(n) - n$, and thus

$$n \text{ perfect} \iff \sigma(n) = 2n.$$

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We factor $945 = 3^3 \cdot 5 \cdot 7$. The divisors of 945 are the numbers $3^a 5^b 7^c$, where $0 \leq a \leq 3, 0 \leq b \leq 1, 0 \leq c \leq 1$.

These are precisely the numbers that show up when we expand

$$(1 + 3 + 3^2 + 3^3)(1 + 5)(1 + 7).$$

So

$$\begin{aligned}\sigma(945) &= (1 + 3 + 3^2 + 3^3)(1 + 5)(1 + 7) \\ &= 40 \cdot 6 \cdot 8 = 1920.\end{aligned}$$

[$\sigma(945) = 1920$ is a little more than $2 \cdot 945$, so 945 is abundant. In fact, 945 is the first odd abundant number.]

In general, if the prime factorization of n has the form $p_1^{e_1} \cdots p_k^{e_k}$, then

$$\begin{aligned}\sigma(n) &= (1 + p_1 + \cdots + p_1^{e_1}) \cdots (1 + p_k + \cdots + p_k^{e_k}) \\ &= \sigma(p_1^{e_1}) \cdots \sigma(p_k^{e_k}).\end{aligned}$$

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A function f from the positive integers to the complex numbers with the property that

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Other examples:

- (1) $f(n) = n$ is multiplicative,
- (2) $h(n) = \sigma(n)/n$ is multiplicative.

Ex. (2) is important for us: n is perfect $\iff h(n) = 2$.

Lemma

No proper multiple of a perfect number is perfect.

Proof.

Suppose n is perfect. We will prove every proper multiple of n is abundant.

Suppose $m = nk$, where $k > 1$. The divisors of m include all the numbers dk , where $d \mid n$. So

$$\begin{aligned}\sigma(m) &\geq k\sigma(n) + 1, \\ &= 2kn + 1 = 2m + 1.\end{aligned}$$

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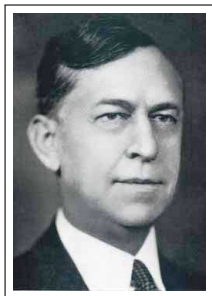
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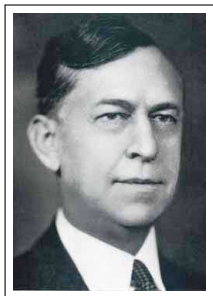
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We will give a **supernatural** proof of Dickson's theorem.

Supernatural numbers

Definition

A **supernatural number** is a formal product

$$2^{e_2} 3^{e_3} 5^{e_5} \dots = \prod_{p \text{ prime}} p^{e_p},$$

where each $e_p \in \{0, 1, 2, 3, \dots\} \cup \{\infty\}$.

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There is a natural notion of what it means for one supernatural number to divide another.

Definition (p -adic valuation)

If N is a supernatural number, and p is a prime, we let $v_p(N)$ be the exponent of p in the factorization of N . Thus, $v_p(N) \in \{0, 1, 2, \dots\} \cup \{\infty\}$.

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Definition (supernatural convergence)

If N_1, N_2, N_3, \dots is a sequence of supernatural numbers, and N is a supernatural number, we say $N_i \rightarrow N$ if:

For every prime p , we have $v_p(N_i) \rightarrow v_p(N)$.

Examples

- $2, 3, 5, 7, 11, 13, \dots$ converges to 1.
- $2, 2^2 \cdot 3^2, 2^3 \cdot 3^3 \cdot 5^3, \dots$ converges to $\prod_p p^\infty$.

Lemma

Every sequence of supernatural numbers has a subsequence that converges to a supernatural number.

Proof.

Exercise! (Related to Tychonoff's theorem.)

For each positive integer k , let \mathcal{S}_k be the set of supernatural numbers where at most k exponents are nonzero.

Lemma

If N_1, N_2, N_3, \dots is a sequence of elements of \mathcal{S}_k converging supernaturally to a limit N . Then $N \in \mathcal{S}_k$.

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Proof.

Let p be a prime dividing N . Say $v_p(N) = e_p$. By definition, $v_p(N_i) = e_p$ for all large i . Choose i large enough that this holds simultaneously for all the (finitely) many primes p dividing N .

Then for all large i , we have $v_p(N) \leq v_p(N_i)$ for all primes p . So $N \mid N_i$.

Recall that $h(N) = \frac{\sigma(N)}{N}$. We can extend $h(N)$ to S_k . How?

If $N \in S_k$, define

$$h(N) = \prod_p h(p^{e_p}).$$

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This is “morally” a finite product.

Here we understand

$$h(p^\infty) = \lim_{e \rightarrow \infty} h(p^e) = \lim_{e \rightarrow \infty} \frac{(p^{e+1} - 1)/(p - 1)}{p^e} = \frac{p}{p - 1}.$$

If N is a natural number with $\leq k$ prime factors, then $h(N)$ makes sense with N thought of as either a natural number, or an element of S_k , and we get the same real number answer.

Lemma (Continuity lemma)

If N_1, N_2, N_3, \dots is a sequence of elements of S_k converging supernaturally to N , then $h(N_i) \rightarrow h(N)$.

Proof of Dickson's theorem.

Suppose for a contradiction that there are infinitely many odd perfect numbers with $\leq k$ distinct prime factors.

Then we can choose a supernaturally convergent sequence of distinct such numbers, say N_1, N_2, N_3, \dots . Say $N_i \rightarrow N$, where

$$N = p_1^{e_1} \cdots p_r^{e_r},$$

where $r \leq k$.

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Each $h(N_i) = 2$, so $h(N) = \lim h(N_i) = 2$.

Observation: At least one of the exponents $e_j = \infty$. Otherwise, N is a natural number, and N divides N_i for all large i . At most one N_i can equal N . So from some point on, N is a proper divisor of N_i , a contradiction!

Write

$$N = p_1^{e_1} \cdots p_r^{e_r},$$

where $r \leq k$ and $h(N) = 2$.

Can order the primes so that $e_1, \dots, e_\ell < \infty$, and $e_{\ell+1}, \dots, e_r = \infty$.

Then

$$2 = \frac{p_1^{e_1+1} - 1}{p_1^{e_1}(p_1 - 1)} \cdots \frac{p_\ell^{e_\ell+1} - 1}{p_\ell^{e_\ell}(p_\ell - 1)} \cdot \frac{p_{\ell+1}}{p_{\ell+1} - 1} \cdots \frac{p_r}{p_r - 1}.$$

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Clear some denominators:

$$\begin{aligned} 2p_1^{e_1} \cdots p_\ell^{e_\ell} (p_{\ell+1} - 1) \cdots (p_r - 1) \\ = \frac{p_1^{e_1+1} - 1}{p_1 - 1} \cdots \frac{p_\ell^{e_\ell+1} - 1}{p_\ell - 1} \cdot p_{\ell+1} \cdots p_r. \end{aligned}$$

Can assume $p_{\ell+1} < \cdots < p_r$. Then p_r divides RHS but not LHS !

Where do we stand today?

After Heath-Brown, Cook, and Nielsen, we have the following explicit forms of Dickson's theorem.

Theorem

If N is odd and perfect with $\leq k$ distinct prime factors, then $N < 2^{4^k}$.

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As a complement to this:

Theorem (P.)

The number of odd perfect N with $\leq k$ distinct prime factors is $< 4^{k^2}$.

Exercises

An **amicable pair** is a pair of integers n, m with $\sigma(n) = \sigma(m) = n + m$. For instance, 220 and 284.

1. Prove that for each k , there are only finitely many amicable pairs n, m with $\Omega(nm) \leq k$. Here $\Omega(a)$ is the sum of the exponents in the prime factorization of a .

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An **amicable pair** is a pair of integers n, m with $\sigma(n) = \sigma(m) = n + m$. For instance, 220 and 284.

1. Prove that for each k , there are only finitely many amicable pairs n, m with $\Omega(nm) \leq k$. Here $\Omega(a)$ is the sum of the exponents in the prime factorization of a .
2. (Harder!) Prove that for each k , there are only finitely many relatively prime amicable pairs n, m with $\omega(nm) \leq k$. Here $\omega(a)$ is the number of distinct prime factors of a .

THANK YOU!