A finiteness theorem for odd perfect numbers

Paul Pollack

University of Georgia

June 23, 2020

Three kinds of natural numbers

> *Among simple even numbers, some are superabundant, others are deficient: these two classes are as two extremes opposed one to the other; as for those that occupy the middle point between the two, they are said to be perfect.*
>
> – Nicomachus (ca. 100 AD), *Introductio Arithmetica*

Let $s(n) = \sum_{d|n, d<n} d$ be the sum of the proper divisors of n.

Abundant: $s(n) > n$.
Deficient: $s(n) < n$.
Perfect: $s(n) = n$.

Three kinds of natural numbers

> *Among simple even numbers, some are superabundant, others are deficient: these two classes are as two extremes opposed one to the other; as for those that occupy the middle point between the two, they are said to be perfect.*
>
> – Nicomachus (ca. 100 AD), *Introductio Arithmetica*

Let $s(n) = \sum_{d|n, d<n} d$ be the sum of the proper divisors of n.

Abundant: $s(n) > n$.
Deficient: $s(n) < n$.
Perfect: $s(n) = n$.

Ex: 5 is deficient ($s(5) = 1$), 12 is abundant ($s(12) = 16$), and 6 is perfect ($s(6) = 6$).

The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.

The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.

. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.

Because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. – Donald Rumsfeld

Theorem (Euclid – Euler)

If $2^n - 1$ is prime, then

$$N := 2^{n-1}(2^n - 1)$$

is perfect. Conversely, if N is an <u>even</u> perfect number, then N has this form.

But what about odd perfect numbers?

Is there a simple formula for odd perfect numbers, like for even perfect numbers? Probably not.

Theorem (Dickson, 1913)

For each positive integer k, there are only finitely many odd perfect numbers with $\leq k$ distinct prime factors.

How do we understand $s(n)$ anyway?

It is more convenient to work with the sum of *all* positive divisors of n, including n itself, which is denoted $\sigma(n)$. That is,

$$\sigma(n) = \sum_{d|n} d.$$

Then $s(n) = \sigma(n) - n$, and thus

$$n \text{ perfect} \iff \sigma(n) = 2n.$$

Example

Compute the sum of all positive divisors of 945 (i.e., $\sigma(945)$).

We factor $945 = 3^3 \cdot 5 \cdot 7$.

Example

Compute the sum of all positive divisors of 945 (i.e., $\sigma(945)$).

We factor $945 = 3^3 \cdot 5 \cdot 7$. The divisors of 945 are the numbers $3^a 5^b 7^c$, where $0 \le a \le 3, 0 \le b \le 1, 0 \le c \le 1$.

These are precisely the numbers that show up when we expand

$$(1 + 3 + 3^2 + 3^3)(1 + 5)(1 + 7).$$

So

$$\begin{aligned} \sigma(945) &= (1 + 3 + 3^2 + 3^3)(1 + 5)(1 + 7) \\ &= 40 \cdot 6 \cdot 8 = 1920. \end{aligned}$$

[$\sigma(945) = 1920$ is a little more than $2 \cdot 945$, so 945 is abundant. In fact, 945 is the first odd abundant number.]

In general, if the prime factorization of n has the form $p_1^{e_1} \cdots p_k^{e_k}$, then

$$\begin{aligned}\sigma(n) &= (1 + p_1 + \cdots + p_1^{e_1}) \cdots (1 + p_k + \cdots + p_k^{e_k}) \\ &= \sigma(p_1^{e_1}) \cdots \sigma(p_k^{e_k}).\end{aligned}$$

In general, if the prime factorization of n has the form $p_1^{e_1} \cdots p_k^{e_k}$, then

$$\begin{aligned}\sigma(n) &= (1 + p_1 + \cdots + p_1^{e_1}) \cdots (1 + p_k + \cdots + p_k^{e_k}) \\ &= \sigma(p_1^{e_1}) \cdots \sigma(p_k^{e_k}).\end{aligned}$$

A function f from the positive integers to the complex numbers with the property that

$$f(n) = f(p_1^{e_1}) \cdots f(p_k^{e_k})$$

is called a **multiplicative function**. So σ is an example of a multiplicative function.

In general, if the prime factorization of n has the form $p_1^{e_1} \cdots p_k^{e_k}$, then

$$\begin{aligned}\sigma(n) &= (1 + p_1 + \cdots + p_1^{e_1}) \cdots (1 + p_k + \cdots + p_k^{e_k}) \\ &= \sigma(p_1^{e_1}) \cdots \sigma(p_k^{e_k}).\end{aligned}$$

A function f from the positive integers to the complex numbers with the property that

$$f(n) = f(p_1^{e_1}) \cdots f(p_k^{e_k})$$

is called a **multiplicative function**. So σ is an example of a multiplicative function.

Other examples:

(1) $f(n) = n$ is multiplicative,

(2) $h(n) = \sigma(n)/n$ is multiplicative.

Ex. (2) is important for us: n is perfect $\iff h(n) = 2$.

Lemma

No proper multiple of a perfect number is perfect.

Proof.

Suppose n is perfect. We will prove every proper multiple of n is abundant.

Suppose $m = nk$, where $k > 1$. The divisors of m include all the numbers dk, where $d \mid n$. So

$$\begin{aligned} \sigma(m) &\geq k\sigma(n) + 1, \\ &= 2kn + 1 = 2m + 1. \end{aligned}$$

So m is abundant.

Lemma

No proper multiple of a perfect number is perfect.

Proof.

Suppose n is perfect. We will prove every proper multiple of n is abundant.

Suppose $m = nk$, where $k > 1$. The divisors of m include all the numbers dk, where $d \mid n$. So

$$\begin{aligned}\sigma(m) &\geq k\sigma(n) + 1,\\ &= 2kn + 1 = 2m + 1.\end{aligned}$$

So m is abundant.

Time to wake up

Theorem (Dickson, 1913)

For each positive integer k, there are only finitely many odd perfect numbers with $\leq k$ distinct prime factors.

Time to wake up

Theorem (Dickson, 1913)

For each positive integer k, there are only finitely many odd perfect numbers with $\leq k$ distinct prime factors.

We will give a **supernatural** proof of Dickson's theorem.

Supernatural numbers

Definition

A **supernatural number** is a formal product

$$2^{e_2} 3^{e_3} 5^{e_5} \cdots = \prod_{p \text{ prime}} p^{e_p},$$

where each $e_p \in \{0, 1, 2, 3, \dots\} \cup \{\infty\}$.

Supernatural numbers

Definition

A **supernatural number** is a formal product

$$2^{e_2}3^{e_3}5^{e_5}\cdots = \prod_{p \text{ prime}} p^{e_p},$$

where each $e_p \in \{0,1,2,3,\dots\} \cup \{\infty\}$.

Examples

1. Every natural number is a supernatural number.
2. $2 \cdot 3^{\infty} \cdot 17$ is also a supernatural number, as is $\prod_p p^{\infty}$.

Supernatural numbers

Definition

A **supernatural number** is a formal product

$$2^{e_2}3^{e_3}5^{e_5}\cdots = \prod_{p \text{ prime}} p^{e_p},$$

where each $e_p \in \{0, 1, 2, 3, \dots\} \cup \{\infty\}$.

Examples

1. Every natural number is a supernatural number.
2. $2 \cdot 3^\infty \cdot 17$ is also a supernatural number, as is $\prod_p p^\infty$.

There is a natural notion of what it means for one supernatural number to divide another.

Definition (p-adic valuation)

If N is a supernatural number, and p is a prime, we let $v_p(N)$ be the exponent of p in the factorization of N. Thus,
$v_p(N) \in \{0, 1, 2, \dots\} \cup \{\infty\}$.

Definition (p-adic valuation)

If N is a supernatural number, and p is a prime, we let $v_p(N)$ be the exponent of p in the factorization of N. Thus,
$v_p(N) \in \{0, 1, 2, \dots\} \cup \{\infty\}$.

Definition (supernatural convergence)

If $N_1, N_2, N_3, \dots$ is a sequence of supernatural numbers, and N is a supernatural number, we say $N_i \to N$ if:

For every prime p, we have $v_p(N_i) \to v_p(N)$.

Examples

- $2, 3, 5, 7, 11, 13, \dots$ converges to 1.
- $2, 2^2 \cdot 3^2, 2^3 \cdot 3^3 \cdot 5^3, \dots$ converges to $\prod_p p^\infty$.

Lemma

Every sequence of supernatural numbers has a subsequence that converges to a supernatural number.

Proof.

Exercise! (Related to Tychonoff's theorem.)

For each positive integer k, let $\mathcal{S}_k$ be the set of supernatural numbers where at most k exponents are nonzero.

Lemma

If $N_1, N_2, N_3, \ldots$ is a sequence of elements of $\mathcal{S}_k$ converging supernaturally to a limit N. Then $N \in \mathcal{S}_k$.

Lemma

If $N_1, N_2, N_3, \ldots$ is a sequence of supernatural numbers converging supernaturally to a limit N. If N is a natural number, then N divides N_i eventually (= for all large i).

Lemma

If $N_1, N_2, N_3, \ldots$ is a sequence of supernatural numbers converging supernaturally to a limit N. If N is a natural number, then N divides N_i eventually (= for all large i).

Proof.

Let p be a prime dividing N. Say $v_p(N) = e_p$. By definition, $v_p(N_i) = e_p$ for all large i. Choose i large enough that this holds simultaneously for all the (finitely) many primes p dividing N.

Then for all large i, we have $v_p(N) \leq v_p(N_i)$ for all primes p. So $N \mid N_i$.

Recall that $h(N) = \frac{\sigma(N)}{N}$. We can extend $h(N)$ to S_k. How?
If $N \in \mathcal{S}_k$, define

$$h(N) = \prod_p h(p^{e_p}).$$

This is "morally" a finite product.

Recall that $h(N) = \frac{\sigma(N)}{N}$. We can extend $h(N)$ to S_k. How?
If $N \in \mathcal{S}_k$, define

$$h(N) = \prod_p h(p^{e_p}).$$

This is "morally" a finite product.

Here we understand

$$h(p^\infty) = \lim_{e\to\infty} h(p^e) = \lim_{e\to\infty} \frac{(p^{e+1}-1)/(p-1)}{p^e} = \frac{p}{p-1}.$$

If N is a natural number with $\leq k$ prime factors, then $h(N)$ makes sense with N thought of as either a natural number, or an element of $\mathcal{S}_k$, and we get the same real number answer.

Lemma (Continuity lemma)

If $N_1, N_2, N_3, \ldots$ is a sequence of elements of S_k converging supernaturally to N, then $h(N_i) \to h(N)$.

Proof of Dickson's theorem.

Suppose for a contradiction that there are infinitely many odd perfect numbers with $\leq k$ distinct prime factors.

Then we can choose a supernaturally convergent sequence of distinct such numbers, say $N_1, N_2, N_3, \ldots$. Say $N_i \to N$, where

$$N = p_1^{e_1} \cdots p_r^{e_r},$$

where $r \leq k$.

Each $h(N_i) = 2$, so $h(N) = \lim h(N_i) = 2$.

Proof of Dickson's theorem.

Suppose for a contradiction that there are infinitely many odd perfect numbers with $\leq k$ distinct prime factors.

Then we can choose a supernaturally convergent sequence of distinct such numbers, say $N_1, N_2, N_3, \dots$. Say $N_i \to N$, where

$$N = p_1^{e_1} \cdots p_r^{e_r},$$

where $r \leq k$.

Each $h(N_i) = 2$, so $h(N) = \lim h(N_i) = 2$.

Observation: At least one of the exponents $e_j = \infty$. Otherwise, N is a natural number, and N divides N_i for all large i. At most one N_i can equal N. So from some point on, N is a proper divisor of N_i, a contradiction!

Write

$$N = p_1^{e_1} \cdots p_r^{e_r},$$

where $r \leq k$ and $h(N) = 2$.

Can order the primes so that $e_1, \ldots, e_\ell < \infty$, and $e_{\ell+1}, \ldots, e_r = \infty$. Then

$$2 = \frac{p_1^{e_1+1} - 1}{p_1^{e_1}(p_1 - 1)} \cdots \frac{p_\ell^{e_\ell+1} - 1}{p_\ell^{e_\ell}(p_\ell - 1)} \cdot \frac{p_{\ell+1}}{p_{\ell+1} - 1} \cdots \frac{p_r}{p_r - 1}.$$

Write

$$N = p_1^{e_1} \cdots p_r^{e_r},$$

where $r \leq k$ and $h(N) = 2$.

Can order the primes so that $e_1, \dots, e_\ell < \infty$, and $e_{\ell+1}, \dots, e_r = \infty$. Then

$$2 = \frac{p_1^{e_1+1} - 1}{p_1^{e_1}(p_1 - 1)} \cdots \frac{p_\ell^{e_\ell+1} - 1}{p_\ell^{e_\ell}(p_\ell - 1)} \cdot \frac{p_{\ell+1}}{p_{\ell+1} - 1} \cdots \frac{p_r}{p_r - 1}.$$

Clear some denominators:

$$2p_1^{e_1} \cdots p_\ell^{e_\ell}(p_{\ell+1} - 1) \cdots (p_r - 1) = \frac{p_1^{e_1+1} - 1}{p_1 - 1} \cdots \frac{p_\ell^{e_\ell+1} - 1}{p_\ell - 1} \cdot p_{\ell+1} \cdots p_r.$$

Can assume $p_{\ell+1} < \cdots < p_r$. Then p_r divides RHS but not LHS !

Where do we stand today?

After Heath-Brown, Cook, and Nielsen, we have the following explicit forms of Dickson's theorem.

Theorem

If N is odd and perfect with $\leq k$ distinct prime factors, then $N < 2^{4^k}$.

Where do we stand today?

After Heath-Brown, Cook, and Nielsen, we have the following explicit forms of Dickson's theorem.

Theorem

If N is odd and perfect with $\leq k$ distinct prime factors, then $N < 2^{4^k}$.

As a complement to this:

Theorem (P.)

The number of odd perfect N with $\leq k$ distinct prime factors is $< 4^{k^2}$.

Exercises

An **amicable pair** is a pair of integers n, m with $\sigma(n) = \sigma(m) = n + m$. For instance, 220 and 284.

1. Prove that for each k, there are only finitely many amicable pairs n, m with $\Omega(nm) \leq k$. Here $\Omega(a)$ is the sum of the exponents in the prime factorization of a.

Exercises

An **amicable pair** is a pair of integers n, m with $\sigma(n) = \sigma(m) = n + m$. For instance, 220 and 284.

1. Prove that for each k, there are only finitely many amicable pairs n, m with $\Omega(nm) \leq k$. Here $\Omega(a)$ is the sum of the exponents in the prime factorization of a.
2. (Harder!) Prove that for each k, there are only finitely many relatively prime amicable pairs n, m with $\omega(nm) \leq k$. Here $\omega(a)$ is the number of distinct prime factors of a.

THANK YOU!