A GENERALIZATION OF THE HARDY–RAMANUJAN INEQUALITY AND APPLICATIONS

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ABSTRACT. Let $\omega(n)$ denote the number of distinct prime factors of the positive integer n. In 1917, Hardy and Ramanujan showed that for all real numbers $x \geq 2$ and all positive integers k,

$$\sum_{\substack{n \leq x \\ \omega(n)=k}} 1 \leq C \frac{x}{\log x} \frac{(\log\log x + D)^{k-1}}{(k-1)!},$$

where C and D are absolute constants. We derive an analogous result when the summand 1 is replaced by $f(n)$, for many nonnegative multiplicative functions f. Summing on k recovers a frequently-used mean-value theorem of Hall and Tenenbaum. We use the same idea to establish a variant of a theorem of Shirokov, concerning multiplicative functions that are $o(1)$ on average at the primes.

1. INTRODUCTION

Write $\omega(n)$ for the number of distinct prime factors of the positive integer n. The inequality of Hardy and Ramanujan referred to in the title is the following estimate, published in 1917 [HR17].

Proposition 1 (Hardy–Ramanujan inequality). *For all real $x \geq 2$ and integers $k \geq 1$,*

$$\sum_{\substack{n \leq x \\ \omega(n)=k}} 1 \ll \frac{x}{\log x} \frac{(\log\log x + O(1))^{k-1}}{(k-1)!}.$$

The implied constants are absolute.

Hardy and Ramanujan were led to Proposition 1 by considering the question

'*How composite is a large random number n?*'

The so-called **Hardy–Ramanujan theorem** provides an answer, taking $\omega(n)$ as a measure of the compositeness of n. That result asserts that for any function $Z = Z(x)$ tending to infinity as $x \to \infty$, we have

$$|\omega(n) - \log\log x| < Z\sqrt{\log\log x}$$

for all but $o(x)$ values of $n \leq x$.

In 1934, Turán gave an easier proof of the Hardy–Ramanujan theorem, by estimating the second moment of $\omega(n) - \log\log x$ [Tur34]. While Turán's argument is undeniably simpler, the original proof based on Proposition 1 gives substantially sharper estimates for the (in)frequency of large deviations of $\omega(n)$ from $\log\log x$. Those stronger bounds are often important in applications.

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This note was prompted by the observation that the method of Hardy and Ramanujan can be adapted to study the partial sums of nonnegative multiplicative functions.

In order to state the main theorem in a generality sufficient for all of our applications, we introduce the notion of a log**-like function**. We say L is log**-like** if $L(x)$ is positive and increasing for $x \geq 2$, and if there is a positive constant K such that, for all $x \geq 2$,

$$\frac{L(x)}{L(x/w)} \leq 1 + K\frac{\log w}{\log x} \qquad \text{whenever } 2 \leq w \leq \min\{\frac{1}{2}x, \sqrt{x}\}. \tag{1}$$

Given our choice of terminology, it is reassuring that $L(x) = \log x$ is log-like: When $w \leq \sqrt{x}$,

$$\frac{\log x}{\log (x/w)} = \frac{1}{1 - \frac{\log w}{\log x}} = 1 + \frac{\log w}{\log x}\left(1 + \left(\frac{\log w}{\log x}\right) + \left(\frac{\log w}{\log x}\right)^2 + \dots\right) \leq 1 + 2\frac{\log w}{\log x},$$

so that (1) holds with $K = 2$. The reader can check that other examples of log-like functions include $L(x) = (\log x)^A$ for any fixed $A > 0$, and $L(x) = \log x \cdot \log\log (2x)$, $L(x) = \log x \cdot \log\log (2x) \cdot \log\log\log (8x)$.

Theorem 2. *Let f be a nonnegative multiplicative function, and let L be a* log*-like function, satisfying* (1) *for the constant K. For all $x \geq 2$, and all positive integers k,*

$$\sum_{\substack{n \leq x \\ \omega(n)=k}} f(n) \leq \frac{Ax}{L(x)} \sum_{j=0}^{k-1} \frac{(\frac{1}{2}BK)^{k-1-j}}{(k-1-j)!} \sum_{\substack{m \leq x \\ \omega(m)=j}} \frac{f(m)}{m}, \tag{2}$$

where

$$A = \sup_{x^{1/2^k} \leq t \leq x} \frac{1}{t/L(t)} \sum_{p^\nu \leq t} f(p^\nu), \quad B = \sup_{x^{1/2^k} \leq t \leq x} \frac{1}{\log t} \sum_{p^\nu \leq t} f(p^\nu) \frac{\log(p^\nu)}{p^\nu}.\ [1]$$

(Here and below, we reserve the letter p for prime numbers.)

Theorem 2 has several pleasant consequences. If we insert the inequality

$$\sum_{\substack{m \leq x \\ \omega(m)=j}} \frac{f(m)}{m} \leq \frac{1}{j!}\left(\sum_{p^\nu \leq x} \frac{f(p^\nu)}{p^\nu}\right)^j$$

into (2) and apply the binomial theorem, we obtain an analogue of Proposition 1 with the summand 1 replaced by $f(n)$.

Corollary 3. *Under the assumptions of Theorem 2,*

$$\sum_{\substack{n \leq x \\ \omega(n)=k}} f(n) \leq \frac{Ax}{L(x)} \cdot \frac{1}{(k-1)!}\left(\sum_{p^\nu \leq x} \frac{f(p^\nu)}{p^\nu} + \frac{1}{2}BK\right)^{k-1}.$$

The original Hardy–Ramanujan inequality can be recovered from Corollary 3 by taking f to be identically 1, taking $L(x) = \log x$ and $K = 2$, and appealing to elementary estimates in prime number theory.

[1]In the definition of A, we understand $\frac{1}{t/L(t)} \sum_{p^\nu \leq t} f(p^\nu)$ as 0 when $t < 2$, even if $L(t)$ is undefined.

Another application of Corollary 3 is a Hardy–Ramanujan inequality for integers composed entirely of primes from a specified set $\mathscr{P}$. Choose f to be the (multiplicative) indicator function of these integers, and take $L(x) = \log x$. Elementary prime number theory shows that the quantities A, B in Theorem 2 are absolutely bounded. We thus deduce that

$$\sum_{\substack{n \le x \\ p \mid n \Rightarrow p \in \mathscr{P} \\ \omega(n) = k}} 1 \ll \frac{x}{\log x} \cdot \frac{1}{(k-1)!} \Bigg(\sum_{\substack{p \in \mathscr{P} \\ p \le x}} \frac{1}{p} + O(1) \Bigg)^{k-1}, \tag{3}$$

where all implied constants are absolute. If we take $\mathscr{P}$ in (3) as the complement of a set of primes $\mathscr{Q}$, we obtain a cross between Proposition 1 and a simple sieve bound:

$$\sum_{\substack{n \le x \\ p \mid n \Rightarrow p \notin \mathscr{Q} \\ \omega(n) = k}} 1 \ll \frac{x}{\log x} \cdot \frac{1}{(k-1)!} \Bigg(\log\log x + O(1) - \sum_{\substack{q \in \mathscr{Q} \\ q \le x}} \frac{1}{q} \Bigg)^{k-1}. \tag{4}$$

The estimate (3) might be contrasted with Theorem 08 of [HT88], which treats the orthogonal situation where the n are unrestricted but one only counts prime factors from $\mathscr{P}$. It would seem surprising if (3) and (4) had not been observed previously, but in any case they do not seem to be well-known.

Summing the estimate (2) of Theorem 2 over all positive integers k, and rearranging, gives

$$\sum_{1 < n \le x} f(n) \le A \frac{x}{L(x)} \exp\left(\frac{1}{2} BK\right) \sum_{m \le x} \frac{f(m)}{m},$$

where

$$A = \sup_{t \le x} \frac{1}{t/L(t)} \sum_{p^\nu \le t} f(p^\nu), \quad B = \sup_{t \le x} \frac{1}{\log t} \sum_{p^\nu \le t} f(p^\nu) \frac{\log(p^\nu)}{p^\nu}.$$

Specializing to $L(x) = \log x$ and $K = 2$, we recover (a slight variant of) a mean-value theorem of Hall and Tenenbaum; compare with Theorem 01 in [HT88]. It should be noted that the proof of our Theorem 2 is decidedly more complicated than the original Hall–Tenenbaum argument; our point with this observation is simply to highlight how Theorem 2 can be thought of a 'dissected' mean-value estimate.

We intend Theorem 2 and Corollary 3 as easy-to-apply, general-purpose upper bounds. But under certain extra conditions, the estimates they yield are asymptotically sharp. We illustrate this by using those results to establish an asymptotic mean-value theorem for certain nonnegative multiplicative functions that are $o(1)$ on average at the primes. Similar results were obtained by Lucht [Luc74] and Shirokov [Shi81], but their methods are quite different.

Theorem 4. *Let f be a nonnegative multiplicative function. Assume that there are positive constants c_1, c_2, with $c_2 < \frac{1}{2}$, such that*

$$f(p^\nu) \le c_1 p^{c_2 \nu} \tag{5}$$

for all primes p and all positive integers ν. Assume further that there is a positive constant A, and a log*-like function $L(x)$ with $L(x)/\log x \to \infty$ as $x \to \infty$, such that*

$$\sum_{p^\nu \le x} f(p^\nu) \sim Ax/L(x). \tag{6}$$

Then

$$\sum_{n\le x} f(n) \sim \frac{Ax}{L(x)} \sum_{m\le x} \frac{f(m)}{m}. \tag{7}$$

Furthermore,

$$\sum_{m\le x} \frac{f(m)}{m} \sim \prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right). \tag{8}$$

In a concluding remark, we sketch an application of Theorem 4 to a question of Loughran [Lou].

2. A dissected mean value theorem: Proof of Theorem 2

When $k=1$, the claim of Theorem 2 is that $\sum_{p^\nu\le x} f(p^\nu) \le Ax/L(x)$.[2] That inequality is clear from the definition of A. So we may assume that $k\ge 2$.

An integer $n\le x$ can have at most one exact prime power divisor exceeding $\sqrt{x}$. Thus, if $\omega(n)=k$, then at least $k-1$ of the exact prime power divisors of n are bounded by $\sqrt{x}$. Consequently, using the notation $P_1, P_2, P_3, \dots$ for prime powers, we have that for $k\ge 2$,

$$\sum_{\substack{n\le x\\ \omega(n)=k}} f(n) \le \frac{1}{k-1} \sum_{P_1\le \min\{\sqrt{x},\frac12 x\}} f(P_1) \sum_{\substack{m\le x/P_1\\ \omega(m)=k-1\\ \gcd(m,P_1)=1}} f(m).$$

If $k\ge 3$, repeating the process on the inner sum yields

$$\sum_{\substack{n\le x\\ \omega(n)=k}} f(n) \le \frac{1}{(k-1)(k-2)} \sum_{\substack{P_1\le \min\{\sqrt{x},\frac12 x\}\\ P_2\le \min\{\sqrt{x/P_1},\frac12 x/P_1\}\\ \gcd(P_1,P_2)=1}} f(P_1)f(P_2) \sum_{\substack{m\le x/P_1P_2\\ \omega(m)=k-2\\ \gcd(m,P_1P_2)=1}} f(m).$$

Continuing in the same way, we eventually find that

$$\sum_{\substack{n\le x\\ \omega(n)=k}} f(n) \le \frac{1}{(k-1)!} \sum_{P_1,\dots,P_{k-1}} f(P_1)\dots f(P_{k-1}) \sum_{\substack{m\le x/P_1\cdots P_{k-1}\\ \omega(m)=1\\ \gcd(m,P_1\cdots P_{k-1})=1}} f(m); \tag{9}$$

here the conditions on the prime powers $P_1,\dots,P_{k-1}$ are that

(i) $P_1,\dots,P_{k-1}$ are pairwise relatively prime,

(ii) each $P_i \le \min\{\sqrt{x/P_1\cdots P_{i-1}}, \frac12 x/P_1\cdots P_{i-1}\}$.

Before proceeding, notice that $x/P_1\ge x^{1/2}$, that $x/P_1P_2 \ge (x/P_1)^{1/2}\ge x^{1/4}$, and in general, $x/P_1\dots P_j \ge x^{1/2^j}$. So from the definition of A, the inner sum on m in the last display is at most $A\frac{x/P_1\cdots P_{k-1}}{L(x/P_1\cdots P_{k-1})}$. (We have ignored the coprimality condition on m,

[2] We take $f(1)=1$ as included in the definition of f being multiplicative.

which would only make the sum smaller.) Moreover,

$$\frac{L(x)}{L(x/P_1\cdots P_{k-1})} = \frac{L(x)}{L(x/P_1)}\frac{L(x/P_1)}{L(x/P_1P_2)}\cdots\frac{L(x/P_1\cdots P_{k-2})}{L(x/P_1\cdots P_{k-1})}$$
$$\leq \left(1+K\frac{\log P_1}{\log x}\right)\left(1+K\frac{\log P_2}{\log(x/P_1)}\right)\cdots\left(1+K\frac{\log P_{k-1}}{\log(x/P_1\cdots P_{k-2})}\right).$$

Revisiting (9) with these estimates in mind, we now see that

$$\sum_{\substack{n\leq x\\ \omega(n)=k}} f(n) \leq \frac{Ax}{(k-1)!\cdot L(x)}\sum_{P_1,\dots,P_{k-1}}\frac{f(P_1)\cdots f(P_{k-1})}{P_1\cdots P_{k-1}}$$
$$\times\left(1+K\frac{\log P_1}{\log x}\right)\left(1+K\frac{\log P_2}{\log(x/P_1)}\right)\cdots\left(1+K\frac{\log P_{k-1}}{\log(x/P_1\cdots P_{k-2})}\right).$$

After multiplying out, the remaining sum on $P_1,\dots,P_{k-1}$ becomes a sum of 2^{k-1} terms, each having the form

$$K^{|\mathcal{I}|}\sum_{P_1,\dots,P_{k-1}}\prod_{i\notin\mathcal{I}}\frac{f(P_i)}{P_i}\prod_{i\in\mathcal{I}}\frac{f(P_i)\log P_i}{P_i\log(x/P_1\cdots P_{i-1})}$$

for some subset $\mathcal{I}\subseteq\{1,2,\dots,k-1\}$.

Let us estimate the contribution from a given index set $\mathcal{I}$. If $\mathcal{I}$ is nonempty, let i_1 be the largest element of $\mathcal{I}$. Fix all P_i with $i\neq i_1$, and consider all choices of P_{i_1} for which $(P_i)_{i=1}^{k-1}$ satisfies (i), (ii) above. In each such tuple, $P_{i_1}\leq\sqrt{x/P_1\cdots P_{i_1-1}}$, and $\sqrt{x/P_1\cdots P_{i_1-1}}\geq x^{1/2^{i_1}}>x^{1/2^k}$. Recalling the definition of B, we deduce that the expression in the last display is bounded above by

$$\left(\frac{1}{2}B\right)K^{|\mathcal{I}|}\sum_{P_i,\ i\neq i_1}\prod_{i\notin\mathcal{I}}\frac{f(P_i)}{P_i}\prod_{\substack{i\in\mathcal{I}\\ i\neq i_1}}\frac{f(P_i)\log P_i}{P_i\log(x/P_1\cdots P_{i-1})}.$$

Here the first sum is over all tuples $(P_i)_{1\leq i\leq k-1,\ i\neq i_1}$ for which there exists some choice of P_{i_1} making $(P_i)_{i=1}^{k-1}$ satisfy (i) and (ii). If $\mathcal{I}\setminus\{i_1\}$ is nonempty, repeat the procedure, working with next largest element of $\mathcal{I}$ in place of i_1. Continuing in this way, we eventually obtain an upper bound of

$$\left(\frac{1}{2}B\right)^{|\mathcal{I}|}K^{|\mathcal{I}|}\sum_{P_i,\ i\notin\mathcal{I}}\prod_{i\notin\mathcal{I}}\frac{f(P_i)}{P_i}.$$

Here the sum is over all tuples $(P_i)_{i\notin\mathcal{I}}$ which can be filled out to a tuple $(P_i)_{i=1}^{k-1}$ satisfying (i) and (ii). For those tuples $(P_i)_{i\notin\mathcal{I}}$, the number $\prod_{i\notin\mathcal{I}}P_i$ is a positive integer not exceeding x with $k-1-|\mathcal{I}|$ distinct prime factors. Furthermore, at most $k-1-|\mathcal{I}|$ such tuples $(P_i)_{i\notin\mathcal{I}}$ give rise to the same product $\prod_{i\notin\mathcal{I}}P_i$ (by unique factorization). So viewing $m=\prod_{i\notin\mathcal{I}}P_i$, we obtain an upper bound of

$$\left(\frac{1}{2}B\right)^{|\mathcal{I}|}K^{|\mathcal{I}|}(k-1-|\mathcal{I}|)!\sum_{\substack{m\leq x\\ \omega(m)=k-1-|\mathcal{I}|}}\frac{f(m)}{m}.$$

Thus, writing j for $|\mathcal{I}|$,

$$\begin{aligned}\sum_{\substack{n\le x\\ \omega(n)=k}} f(n) &\le \frac{Ax}{(k-1)!\cdot L(x)}\sum_{\mathcal{I}}\left(\frac{1}{2}B\right)^{|\mathcal{I}|} K^{|\mathcal{I}|}(k-1-|\mathcal{I}|)! \sum_{\substack{m\le x\\ \omega(m)=k-1-|\mathcal{I}|}} \frac{f(m)}{m}\\ &= \frac{Ax}{(k-1)!\cdot L(x)}\sum_{j=0}^{k-1}\binom{k-1}{j}\left(\frac{1}{2}BK\right)^{j}(k-1-j)! \sum_{\substack{m\le x\\ \omega(m)=k-1-j}} \frac{f(m)}{m}\\ &= \frac{Ax}{L(x)}\sum_{j=0}^{k-1}\frac{\left(\frac{1}{2}BK\right)^{j}}{j!} \sum_{\substack{m\le x\\ \omega(m)=k-1-j}} \frac{f(m)}{m}.\end{aligned}$$

Replacing j with $k-1-j$ gives Theorem 2.

3. Multiplicative functions with average $o(1)$ at the primes: Proof of Theorem 4

We begin at the end, with a proof of the asymptotic relation (8).

We can view (8) as the claim that, as $x\to\infty$,

$$\sum_{\substack{m>x\\ p|m\Rightarrow p\le x}} \frac{f(m)}{m} = o\left(\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right)\right).$$

The condition (5) on the size of $f(p^\nu)$ implies that the right-hand product on p is $\asymp_f \exp(\sum_{p\le x}\frac{f(p)}{p})$. So our task is to show that

$$\sum_{\substack{m>x\\ p|m\Rightarrow p\le x}} \frac{f(m)}{m} = o\left(\exp\left(\sum_{p\le x}\frac{f(p)}{p}\right)\right). \tag{10}$$

We apply Rankin's trick. Fix $u>0$. For all large x,

$$\begin{aligned}\sum_{\substack{m>x\\ p|m\Rightarrow p\le x}} \frac{f(m)}{m} &\le \sum_{\substack{m\ge 1\\ p|m\Rightarrow p\le x}} \frac{f(m)}{m}\left(\frac{x}{m}\right)^{-u/\log x}\\ &= \exp(-u)\sum_{\substack{m\ge 1\\ p|m\Rightarrow p\le x}} \frac{f(m)}{m^{1-u/\log x}} \ll_f \exp(-u)\exp\left(\sum_{p\le x}\frac{f(p)}{p^{1-u/\log x}}\right).\end{aligned}$$

Now $p^{u/\log x} = 1+O_u(\log p/\log x)$ for $p\le x$. Thus,

$$\sum_{p\le x}\frac{f(p)}{p^{1-u/\log x}} \le \sum_{p\le x}\frac{f(p)}{p} + O_u\left(\frac{1}{\log x}\sum_{p\le x}\frac{f(p)\log p}{p}\right).$$

Keeping in mind that $L(x)/\log x\to\infty$ as $x\to\infty$, the asymptotic relation (6) implies, by partial summation, that

$$\sum_{p^\nu\le x}\frac{f(p^\nu)\log p^\nu}{p^\nu} = o(\log x), \tag{11}$$

as $x \to \infty$. So the expression inside our last O_u-term tends to 0, which implies that $\exp(\sum_{p\le x} \frac{f(p)}{p^{1-u/\log x}}) \le (1+o(1))\exp(\sum_{p\le x}\frac{f(p)}{p})$, as $x\to\infty$. Collecting our estimates, we see that for all large x,

$$\sum_{\substack{m>x\\ p\mid m\Rightarrow p\le x}} \frac{f(m)}{m} \ll_f \exp(-u)\exp\left(\sum_{p\le x}\frac{f(p)}{p}\right).$$

Since u can be taken arbitrarily large, (10), and also (8), follows.

Before proceeding with the proof of (7) we make the following observation: For each fixed $\epsilon>0$,

$$\sum_{m\le x^{\epsilon}}\frac{f(m)}{m} \sim \sum_{m\le x}\frac{f(m)}{m}, \tag{12}$$

as $x\to\infty$. The proof will use, again, that (6) holds with an $L(x)$ of larger order than $\log x$; note in particular that $f=1$ does *not* satisfy (10) or (12). In view of what we showed above, (12) will be proved if

$$\prod_{x^{\epsilon}<p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right) = 1+o(1).$$

Now the left-hand side is here at least 1, and at most

$$\exp\left(\sum_{j\ge 1}\sum_{x^{\epsilon}<p\le x}\frac{f(p^j)}{p^j}\right).$$

The bound (5) suffices to show that the terms with $j\ge 2$ make a total contribution to the double sum of $o(1)$, as $x\to\infty$. The terms with $j=1$ also contribute $o(1)$, as we see by partial summation applied to (6). The relation (12) follows.

We now commence the proof of (7). We begin with the lower bound.

We consider the contribution to $\sum_{n\le x} f(n)$ from the values $n=pm$, where $m\le x^{1/3}$ and $x^{1/2}<p\le x/m$. Note that p and m are relatively prime, and that distinct choices of p and m give rise to distinct products pm. For each $m\le x^{1/3}$, the terms pm contribute

$$f(m)\sum_{x^{1/2}<p\le x/m} f(p).$$

Now

$$\sum_{x^{1/2}<p\le x/m} f(p) = \sum_{x^{1/2}<p^{\nu}\le x/m} f(p^{\nu}) - \sum_{\substack{x^{1/2}<p^{\nu}\le x/m\\ \nu\ge 2}} f(p^{\nu}).$$

Each term in the subtracted sum has size at most $c_1(x/m)^{c_2}$, and there are $O((x/m)^{1/2})$ terms; thus, this sum has size $O((x/m)^{\frac12+c_2})$. Hence,

$$\sum_{x^{1/2}<p\le x/m} f(p) = (1+o(1))\frac{Ax/m}{L(x/m)} - (1+o(1))\frac{Ax^{1/2}}{L(x^{1/2})} - O((x/m)^{\frac12+c_2}). \tag{13}$$

Here the $o(1)$ notation indicates decay to 0 as $x\to\infty$, uniformly in $m\le x^{1/3}$. Next, observe that

$$\frac{L(x)}{L(x^{1/2})} \le 1+\frac{K}{2} \qquad \text{for all } x\ge 4,$$

which implies that $L(x) \le (\log x)^{O(1)}$ for large x. Keeping in mind that $m \le x^{1/3}$, we now see that the first term being subtracted in (13) is of smaller order than $\frac{Ax/m}{L(x/m)}$. The same is true for the second subtracted term. We deduce that

$$\begin{aligned} f(m) \sum_{x^{1/2} < p \le x/m} f(p) &\ge (1+o(1)) \frac{Ax}{L(x/m)} \frac{f(m)}{m} \\ &\ge (1+o(1)) \frac{Ax}{L(x)} \frac{f(m)}{m}. \end{aligned}$$

Summing on $m \le x^{1/3}$, we conclude that

$$\sum_{n \le x} f(n) \ge (1+o(1)) \frac{Ax}{L(x)} \sum_{m \le x^{1/3}} \frac{f(m)}{m}.$$

The lower bound half of (7) now follows from (12).

Turning to the upper bound, we first handle the contribution from n having $\omega(n) \le \log\log x$. Fix $\epsilon > 0$. When $k \le \log\log x$,

$$x^{1/2^k} = \exp(\frac{1}{2^k} \log x) \ge \exp((\log x)^{1-\log 2}),$$

which tends to infinity with x. Thus, for all large x, and every $k \le \log\log x$,

$$\sup_{x^{1/2^k} \le t \le x} \frac{1}{t/L(t)} \sum_{p^\nu \le t} f(p^\nu) \le (1+\epsilon)A,$$

and

$$\sup_{x^{1/2^k} \le t \le x} \frac{1}{\log t} \sum_{p^\nu \le t} f(p^\nu) \frac{\log(p^\nu)}{p^\nu} \le \epsilon.$$

(To obtain the second of these estimates, recall (11).) Summing (2) on k (with A replaced by $(1+\epsilon)A$ and B by ϵ), we deduce that for all large enough x,

$$\begin{aligned} \sum_{\substack{1 < n \le x \\ \omega(n) \le \log\log x}} f(n) &\le \sum_{k \le \log\log x} \left(\frac{A(1+\epsilon)x}{L(x)} \sum_{j=0}^{k-1} \frac{(\frac{1}{2}\epsilon K)^{k-1-j}}{(k-1-j)!} \sum_{\substack{m \le x \\ \omega(m) = j}} \frac{f(m)}{m} \right) \\ &\le A(1+\epsilon) \cdot \frac{x}{L(x)} \exp\left(\frac{1}{2}\epsilon K\right) \sum_{m \le x} \frac{f(m)}{m}. \end{aligned}$$

Since ϵ can be taken arbitrarily small, we have an upper bound of the correct shape for the contribution from these restricted values of n.

It therefore suffices to show that those n with $\omega(n) > \log\log x$ make a contribution of lower order. For this we use Corollary 3. We can pick constants A' and B' such that, for all $t \ge 2$,

$$\frac{1}{t/L(t)} \sum_{p^\nu \le t} f(p^\nu) \le A' \quad \text{and} \quad \frac{1}{\log t} \sum_{p^\nu \le t} f(p^\nu) \frac{\log p^\nu}{p^\nu} \le B'.$$

By Corollary 3,

$$\sum_{\substack{n \le x \\ \omega(n) > \log\log x}} f(n) \le \frac{A'x}{L(x)} \sum_{k > \log\log x} \frac{1}{(k-1)!} \left(\sum_{p^\nu \le x} \frac{f(p^\nu)}{p^\nu} + \frac{1}{2} B' K \right)^{k-1}.$$

By (6) and partial summation, the sum of $f(p^\nu)p^{-\nu}$ over $p^\nu \le x$ is $o(\log\log x)$. This implies that the sum on k is dominated, in magnitude, by its first term. Using Stirling's formula to estimate that term, we see that this sum on k is $o(1)$ (in fact, decaying to 0 faster than any negative power of $\log x$), and so is certainly $o(\sum_{m\le x} f(m)/m)$. This completes the proof of Theorem 4.

Remark. On MathOverflow, Loughran [Lou] asked whether there exists a $c > 0$ and a nonnegative multiplicative function f for which

$$\sum_{n\le x} f(n) \sim c\frac{x}{\log x}, \tag{14}$$

as $x \to \infty$. His question was answered in the affirmative by Quas (ibid.), who exhibited such an f explicitly. Quas's example is somewhat pathological, being supported on the very sparse set of integers having the form $2^k 3^\ell$ for nonnegative integers k, ℓ. Of course, in order for (14) to hold for an f of such sparse support, f must take on arbitrarily large values. We now explain how Theorem 4 gives us an f satisfying (14) with each $f(n) \in \{0, 1\}$.

Recall that the sequence of **Golomb primes**, introduced in [Gol55], is defined as follows. We let $p_1 = 3$, and we inductively define p_k as the smallest prime exceeding p_{k-1} that is not congruent to 1 modulo p_i for any $i < k$. We write $\mathscr{P}$ for the set of Golomb primes. With $\pi_{\mathscr{P}}(x)$ the counting function of $\mathscr{P}$, Erdős proved in [Erd62] that

$$\pi_{\mathscr{P}}(x) \sim \frac{x}{\log x \cdot \log\log x}$$

as $x \to \infty$ (see the theorem on p. 1 of [Erd62]), and also (eq. (37) of [Erd62]) that

$$\prod_{\substack{p\in\mathscr{P}\\ p\le x}} \left(1 - \frac{1}{p-1}\right) \sim \frac{1}{\log\log x}.$$

Let f be the characteristic function of those integers composed entirely of primes from $\mathscr{P}$. Inserting Erdős's estimates into Theorem 4 (applied with $L(x) = \log x \cdot \log\log(2x)$), it is straightforward to derive that

$$\sum_{n\le x} f(n) \sim cx/\log x, \quad \text{where} \quad c = \prod_{p\in\mathscr{P}} \left(1 - \frac{1}{(p-1)^2}\right).$$

(In this argument, we could have applied [Luc74, Satz 3] or [Shi81, Theorem 2] in place of our Theorem 4.)

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