# The distribution of abundant numbers

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# All Greek to us

*Among simple even numbers, some are superabundant, others are deficient: these two classes are as two extremes opposed one to the other; as for those that occupy the middle point between the two, they are said to be perfect.*

– Nicomachus (ca. 100 AD),

Let $s(n) := \sum_{d|n,\ d<n} d$ be the sum of the **proper** divisors of $n$.

**Abundant:** $s(n) > n$, e.g., $n = 12$.
**Deficient:** $s(n) < n$, e.g., $n = 5$.
**Perfect:** $s(n) = n$, e.g., $n = 6$.

# More wisdom of the ancients



*The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.*

*. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.*

# You can see a lot just by looking

Let's list the first several terms of each of these sequences.

**Abundants:** 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, ....

**Deficients:** 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, ....

**Perfects:** 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, ....

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**Perfects:** 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, . . . .

> *Just as . . . ugly and vile things abound, so superabundant and deficient numbers are plentiful and can be found without a rule. . . – Nicomachus*

# Making sense of nonsense

If $A$ is a subset of $\mathbb{N} = \{1, 2, 3, \dots\}$, define the *density* of $A$ as

$$\lim_{x \to \infty} \frac{\# A \cap [1, x]}{x}.$$

For example, the even numbers have density $1/2$, and the prime numbers have density $0$. But the set of natural numbers with first digit $1$ does not have a density.

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**Question (Bessel-Hagen, 1929)**

*Does the set of abundant numbers have a density? What about the deficient numbers? The perfect numbers?*

# It's OK to be dense

## Theorem (Davenport, 1933)

*For each real $u \geq 0$, consider the set*

$$\mathcal{D}_s(u) = \{n : s(n)/n \leq u\}.$$

*This set has an asymptotic density $D_s(u)$. As a function of $u$, the function $D_s$ is continuous with $D_s(0) = 0$ and $D_s(\infty) = 1$. Moreover (Schoenberg) $D_s$ is strictly increasing for $u \geq 0$.*

## Corollary

*The perfect numbers have density $0$, the deficient numbers have density $D_s(1)$, and the abundant numbers have density $1 - D_s(1)$.*

# Asking the easy questions: Someone's got to do it

So if we let $A(x)$ denote the number of abundant numbers not exceeding $x$, then

$$A(x) = (1 - D_s(1))x + E(x),$$

where the error term

$$E(x) = o(x), \quad \text{as } x \to \infty.$$

**Question (What's really going on with the main term?)**

*What is the constant $1 - D_s(1)$, numerically?*

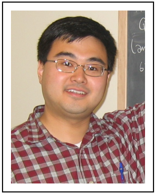
**Question (What's really going on with the error term?)**

*How large is $E(x)$? Sure, it's $o(x)$, but how big a $o(x)$ is it?*

# The main term

The coefficient of $x$ has received a large amount of study. In fact, even before Davenport had proved the existence of the density, Behrend had studied the upper and lower densities for his Ph.D. thesis in the early 1930s.

The following theorem improves on earlier work of Behrend, Salié, Wall, and Deléglise:



**Theorem (Kobayashi, 2010)**

*For the density of abundant numbers, we have*

$$0.24761 < 1 - D_s(1) < 0.24765.$$

*So just under $1$ in every $4$ natural numbers is abundant, and just over $3$ in $4$ are deficient.*

# OK, what about the error term?

Why ask a specific question when you can ask a general one? Let's define the counting function of the $u$-nondeficient numbers by

$$A(u;x) := \#\{n \leq x : s(n)/n \geq u\},$$

and let

$$E(u;x) := A(u;x) - (1 - D_s(u))x.$$

**Question**

*How big is $E(u;x)$?*

The original problem corresponds to $u = 1$. (Actually $A(u;x)$ also throws in perfect numbers, but this is negligible.)

# Does everyone love an estimate in uniform?

**Theorem (Elliott/Fainleib, 1968)**

*For all* $u > 0$ *and all* $x \geq e^{e^e}$,

$$E(u;x) \ll \frac{x}{\log x}\left(\frac{\log\log x}{\log\log\log x}\right).$$

*The implied constant here is absolute.*

If $u$ may depend on $x$, this estimate is almost best possible; it is not hard to show that

$$E(1+\frac{1}{\log x};x) \gg \frac{x}{\log x}.$$

But what if $u$ is **fixed**? For example, what if $u = 1$ and $x \to \infty$?

# Does everyone love an estimate in uniform?



The overarching idea of Fainleib's proof is to apply a variant of the Berry–Esseen theorem from probability to to measure the distance between the discrete distribution

$$D_s(u; x) := \frac{1}{\lfloor x \rfloor} \#\{n \le x : s(n)/n \ge u\}$$

and the Davenport distribution function $D_s(u)$.

- The proof is more analytic than arithmetic; uses characteristic functions (Fourier transforms).
- This Berry–Esseen variant 'sees' all $u$ at once.

# Our results



Our goal: Get an improved estimate when $u$ is fixed, sticking to mostly 'elementary' tools (sieve methods, distribution of smooth numbers, etc.).

In our paper, we recover Fainleib's estimate using only elementary tools. And we obtain a modest improvement in the case of fixed $u$.

**Theorem (Kobayashi, P.)**

*Fix a real number $u > 0$. Then as $x \to \infty$,*

$$E(u;x) = o\left(\frac{x}{\log x}\left(\frac{\log\log x}{\log\log\log x}\right)\right).$$

# Our results

In fact, for 'most' values of $u$, we can do much better than this.

## Definition

A real number $u$ is called a *Liouville number* if (1) $u$ is irrational, and (2) for every $\kappa > 0$, there is a pair of integers $p$, $q$ with $q > 1$ satisfying

$$\left|\frac{p}{q} - u\right| < \frac{1}{q^\kappa}.$$

It is an exercise in graduate analysis to show that the set of Liouville numbers has Lebesgue measure zero.

# Our results



**Theorem (Kobayashi and P.)**

*Suppose that $u > 0$ is not a Liouville number. For a certain constant $\beta = \beta(u) > 0$ and all $x > x_0(u)$,*

$$|E(u;x)| < x\exp(-\beta(\log x)^{1/3}(\log\log x)^{2/3}).$$

In particular, this error estimate holds in the case $u = 1$.

# Some components of the proofs

Both proofs rely on the theory of primitive abundant numbers.

**Definition (due to Erdős)**

Let $u > 0$. A natural number $n$ is called *primitive $u$-abundant* if $\frac{s(n)}{n} \geq u$ while $\frac{s(d)}{d} < u$ for every proper divisor $d$ of $n$.

It is easy to prove that a number $n$ satisfies $\frac{s(n)}{n} \geq u$ exactly when it possesses a primitive $u$-abundant divisor. So we can write

$$\{n \leq x : \frac{s(n)}{n} \geq u\} = \bigcup_{d} \{n \leq x : d \mid n\},$$

where $d \leq x$ runs through primitive $u$-abundant numbers.

# Proofiness, ctd.

**Theorem (Kobayashi)**

*To each $u$-nondeficient $n$, one can associate a canonical primitive $u$-abundant divisor $d$ of $n$.*

This lets us write the $u$-nondeficient numbers as a *disjoint union*:

$$\{n \le x : \frac{s(n)}{n} \ge u\} = \dot{\bigcup_d} \{n \le x : n \text{ is canonically associated to } d\},$$

So to estimate the left-hand side, we can just add the sizes of the sets that appear on the right.

# Proofiness, ctd.

The definition of $d$ is such that the statement

$n$ possesses $d$ as its canonical primitive $u$-abundant divisor

is equivalent to saying that $n = dm$, where $m$ satisfies some coprimality conditions. So to count the number $n$, we can apply sieve methods! (Specifically, we use a version of the "fundamental lemma".)

Finally, we use results of Erdős about the distribution of primitive abundants, such as the fact that

$$\#\{d \le x : d \text{ primitive } u\text{-abundant}\} \le x/\exp(\eta\sqrt{\log x \log\log x})$$

if $u$ is a fixed, non-Liouville number; here $\eta = \eta(u) > 0$.

# Open questions



**Question**

*Let's suppose* $u = 1$ *for simplicity. What is the true maximal order of* $E(1;x)$*? Can one decide whether or not* $E(1;x) = O(x^{1-\epsilon})$*? Probably it is not the case that* $E(1;x) = O(\log x)$*, but we do not see how to disprove this!*

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**Question**

*Let* $A(x; q, a)$ *be the number of abundants not exceeding* $x$ *belonging to the progression* $a \pmod q$. *It is known that* $A(x; 4, 1) \sim A(x; 4, 3)$ *as* $x \to \infty$.

*Can the difference* $A(x; 4, 1) - A(x; 4, 3)$ *be arbitrarily large? Does this difference change sign infinitely often?*

# THANK YOU!