Excursions in Number Theory, Algebra, and Analysis

By Kenneth Ireland and Al Cuoco

Reviewed by Paul Pollack

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I first heard Kenneth Ireland's name in the summer of 1998. At that time I was a rising high school senior attending an eight-week mathematics summer camp at Ohio State University. Arnold Ross had founded this program in the 1950s in order to provide a foundation in abstract mathematics to highly-motivated mathematically and scientifically inclined youngsters. Early on Ross had decided to center the program around elementary number theory and its connections with modern algebra, viewing these topics as both inviting and accessible to mathematical neophytes. Textbooks were verboten for first-year campers — Arnold Ross was opposed to students having access to anything that might "spoil their fun" — but Ireland and Rosen's Springer GTM ('A classical introduction to modern number theory') enjoyed semiofficial status as the book one was supposed to read after the program was over.

After the camp I did as I was told. And in the years since, I spent many enjoyable hours poring over Ireland and Rosen's text. I was fortunate to meet Michael Rosen during my graduate student years, but Kenneth Ireland — who he was, and what had prepared him to coauthor such a magnificent text — was always something of a mystery.

It was my good luck then to receive the invitation to review *Excursions in Number Theory, Algebra, and Analysis*. The book is a remarkable tribute to the legacy of Kenneth Ireland. The preface features brief reminiscences of Ireland as a man and mathematician from Michael Rosen, Ken Ribet, and colleagues at New Brunswick. In the remainder of the manuscript, we get a look at heretofore-unseen course notes of a highly gifted expositor, touched up with loving care by one of his students (Cuoco).

The first thing prospective readers should be aware of is that this is "not your grandmother's mathematics text" (in the words of the authors). The unusual structure is explained by an unusual origin. These notes were designed by Ireland for a 1972 six-week, residential teacher-training program offered at Bowdoin College. In common with a traditional textbook, most chapters contain theorems, proofs, discussions, and exercises. But those chapters are not intended as the primary point of contact with the text! Instead, the beating heart of the manuscript is a series of 200 exercises, broken into eight sets and spread across the fourteen pages of Chapter 1, termed "Dialing-in Problems". We are told emphatically ("I can't stress enough") that Chapters 2 and on "exist to support [one's] work on these problems."

Ireland and Cuoco are firm believers that students should be working examples, noticing patterns, and building intuition before being told statements or proofs of general theorems. In a standard textbook, stating a theorem in a form sufficient to encompass all later applications is seen as a hallmark of efficient exposition. Particularly appealing special cases are noted only after the entire logical edifice has been constructed. The Dialing-In Problems reverse this structure. Special cases appear as objects worthy of respect in their own right. They function to excite curiosity and conjecture. When that excitement reaches a boil, Chapters 2–6 are available as a resource for those ready to appreciate elegant general statements. Cuoco refers to this framework as "experience before formality", though I may prefer the snappier phrase (also mentioned in the text) "exposure before closure".

That's the *philosophy* behind the book. What about the *mathematics*? Kenneth Ireland was a number theorist, and so it is perhaps not surprising that number theory takes center stage in this course. Important ideas from algebra and analysis appear, but they play a supporting rather than a starring role.

Here is a quick run-down of the contents, beginning with Chapter 2. (Skip to the end of the summary for a few comments on the Dialing-In Problems, which make up Chapter 1.)

Chapter 2, bearing the innocent-enough title "Polygons and modular arithmetic," is the longest (36 pages) and perhaps most far-ranging of the entire book. It opens with the construction of $\mathbb{C}$ as the collection of ordered pairs of real numbers, endowed with appropriate definitions of $+$ and $\cdot$. It ends with a discussion of group actions, with Wielandt's simple proof of Sylow's first theorem being a notable highlight. Along the way we find motivated statements of the Kronecker–Weber theorem and Gauss's law of quadratic reciprocity, as well as some hints as to why the regular heptagon is not constructible by straightedge and compass. (The corresponding *proofs* lie outside the scope of the book, but enough is said here to give readers an honest appreciation of the results.) A fair bit of algebra is developed in the course of these discussions, including Eisenstein's irreducibility criterion and the cyclicity of finite subgroups of $F^\times$ (F a field).

Chapter 3 is centered around the concept of unique factorization. The fundamental theorem of arithmetic is proved in §3.1, starting from the division algorithm; the argument is standard, although the details are carried out more concretely here than in many other sources. In §3.2 it is noted that the same chain of reasoning used to develop unique factorization in $\mathbb{Z}$ applies equally well in the ring $\mathbb{Z}[i]$ of Gaussian integers. Section 3.3 shows how unique factorization in $\mathbb{Z}[i]$ leads to an elegant proof of "Fermat's Christmas theorem," characterizing those primes that can be written as sums of two integer squares. Section 3.4 is a gem; we find here a lucid discussion of formal Dirichlet series. This topic is rarely featured in textbooks (and when it is discussed, usually under the name "Dirichlet convolution," it is often mutilated beyond recognition). The utility of these ideas is illustrated with a proof of Jacobi's gorgeous formula for

$$r_2(n) = \#\{(x,y) \in \mathbb{Z}^2 : x^2 + y^2 = n\},$$

namely

$$r_2(n) = \sum_{\substack{d \mid n \\ d \text{ odd}}} (-1)^{(d-1)/2}.$$

Section 3.5 continues with the theme of sums of squares, reviewing some of what is known and still unknown around Hilbert's 17th problem. This last section does a fantastic job of illustrating how a worthy research problem continues to provide inspiration and challenges long after it is solved.

Chapter 4 revolves around the Fundamental Theorem of Algebra. Four proofs are discussed. The first is a direct analytic argument, along lines proposed by Argand in the early 19th century: If $f(z)$ is a nonconstant polynomial, then $|f(z)|$ assumes a global minimum at some $z_0 \in \mathbb{C}$. But if $|f(z_0)| \neq 0$, then wiggling z_0 a bit one locates a z with $|f(z)| < f(z_0)|$. Absurd! The second proof is almost pure algebra; the deepest input needed from analysis is that polynomials of odd degree over $\mathbb{R}$ have a real root, which is immediate from the intermediate value theorem. The essential algebraic ingredients are the Fundamental Theorem of Symmetric Functions and the existence of abstract splitting fields, both of which receive an elegant treatment here. The third proof, due to Artin and Schreier, is closely related to the second but eliminates symmetric function theory in favor of Galois theory. The fourth proof is topological, based around an 'intuitive' treatment of the the concept of the winding number. A concluding section of the chapter develops the basic theory of cyclotomic polynomials, proving that all their coefficients lie in $\mathbb{Z}$ and that all these polynomials are irreducible over $\mathbb{Q}$. This last section seems a slightly awkward fit to me. I suspect the authors were seeking an excuse to discuss the coefficients of the cyclotomic polynomials, which have been a wellspring of conjectures over the years.

Chapter 5 discusses irrationality and transcendence. Liouville's theorem on approximating algebraic numbers by rationals is proved in §5.1 and used to explicitly construct uncountably many transcendental numbers. Section 5.2 is devoted to carefully stating (without proof) two monumental theorems, the Lindemann–Weierstrass theorem and the Gelfond–Schneider theorem. The irrationality of e is proved in §5.3, as a consequence of the bounds $0 < n!\left|e - \sum_{k=0}^{n} \frac{1}{k!}\right| < \frac{1}{n}$. While simple (and well-known), this argument serves to illustrate the utility of the trivial seeming proposition that there are no integers between 0 and 1 — a principle that finds repeated application later in the chapter. Proofs for the irrationality of π and of e^c, for nonzero integers c, appear in §5.4. The transcendence of e is proved in §5.5 while the transcendence of π is proved in §5.6. While the authors' treatment of these results is eminently readable, I have to confess that despite many years of 'exposure therapy' these proofs still seem to me somewhat mysterious.

The final chapter of the book is dedicated to the rudiments of Fourier series, culminating in a proof of Dirichlet's pointwise convergence theorem: *If f is continuous on $[-\pi,\pi]$ and differentiable at $x_0 \in (-\pi,\pi)$, then f is represented by its Fourier series at x_0.* While the proof proceeds along standard lines (recognizing the partial sums of the Fourier series as a convolution with the Dirichlet kernel), the presentation is unusually down-to-earth. In particular, the reader is not assumed to know the meaning of the symbols "L^1" or "L^2." The authors illustrate the relevance of Fourier series to number theory by presenting in detail Dirichlet's method for evaluating Gauss's sums $\sum_{k=0}^{n-1} \mathrm{e}^{2\pi i k^2/n}$.

Chapters 2–6 are self-contained, with their own sets of exercises. But for maximal benefit, readers should have spent time with the Dialing-in Problems and must possess a willingness to revisit them. Many of the Dialing-in Problems ask readers to work out motivating examples: Find generators of the multiplicative group modulo this prime number p, compute the greatest common divisor of these two Gaussian integers, express this particular polynomial in terms of elementary symmetric functions, etc. Others ask the reader to do theoretical brush-clearing, establishing simple facts needed for later, more complicated arguments. The Dialing-in Problems also frequently refer back to material covered in the lectures (chapters), tasking the reader with writing out a proof of a general theorem in a special case, or challenging them as to why an argument couldn't have been done in a seemingly simpler way. In each case, the intent is to encourage the development of mental habits that more seasoned mathematicians take for granted.

The writing here is first rate. The prose strikes a rare balance between informality and precision that makes the book a joy to read. Furthermore, not only are the topics chosen of intrinsic interest (at least in this reviewer's opinion), but the authors never miss an opportunity to to call attention to related tidbits of beautiful mathematics. For instance, the multiplicativity of the complex absolute value is placed alongside Hurwitz's 1-2-4-8 theorem on products of sums of squares. And as soon as $\zeta(2)$ is evaluated, by an elegant method of Giesky (new to this reviewer), readers are treated to a discussion of why $\zeta(2)^{-1}$ represents the probability two 'randomly chosen' integers are relatively prime. Examples of this kind could be multiplied endlessly. The end result is a portrait of mathematics as a coherent whole that is downright inspiring.

It should be abundantly clear by now that I'm a big fan of this text. For a student at the right age and stage, encountering the book within the right context could change their entire outlook on mathematics.

It's worth some thought as to how someone like me, who works within the university, can provide that context. Certainly bits and pieces of the text could be extracted for seminar or topics courses. I can easily imagine students presenting the theorems from Chapter 5 as part of a reading course on irrationality and transcendence. But teaching a class in the spirit of Ireland's 1972 course would seem significantly more difficult. It is suggested in the preface that the text be used for a senior capstone course, and I approve of this idea heartily, *in principle*. However, even if one is at a university which offers such courses (mine does not), it does not seem easy to find students who fit the profile Ireland and Cuoco have targeted. For instance, I suspect that someone who hasn't seen any ring theory would find themselves quickly overwhelmed, but someone with a semester of algebra under their belt would see much of Chapters 2 and 3 (and many of the associated Dialing-In Problems) as old hat. Difficulties of this sort can be worked around by a sufficiently invested instructor. Nevertheless, it's important to acknowledge that anyone wishing to center a course around this text will face significant challenges. Given all that is on offer here, these challenges would seem well worth taking on!