Uniform Distribution Theory **6** (2011), no.2, 000–000

uniform distribution theory

EQUIDISTRIBUTION MOD q OF ABUNDANT AND DEFICIENT NUMBERS

Paul Pollack

ABSTRACT. The ancient Greeks called the natural number m *deficient*, *perfect*, or *abundant* according to whether $\sigma(m) < 2m$, $\sigma(m) = 2m$, or $\sigma(m) > 2m$. In 1933, Davenport showed that all three of these sets make up a well-defined proportion of the positive integers. More precisely, if we let

$$\mathscr{D}(u;x) := \left\{m \le x : \frac{m}{\sigma(m)} \le u\right\}, \quad \text{and put} \quad D(u;x) := \#\mathscr{D}(u;x),$$

then Davenport's theorem asserts that $\lim_{x\to\infty} \frac{1}{x} D(u;x)$ exists for every u. Moreover, $D(u)$ is a continuous function of u, with $D(0) = 0$ and $D(1) = 1$. In this note, we study the distribution of $\mathscr{D}(u;x)$ in arithmetic progressions. A simple to state consequence of our main result is the following: Fix $u \in (0,1]$. Then the elements of $\mathscr{D}(u;x)$ approach equidistribution modulo prime numbers q whenever q, x, and $\frac{x}{q \log\log\log x}$ all tend to infinity.

Communicated by

1. Introduction

Recall that the natural number m is said to be *deficient* if $\sigma(m) < 2m$ (for example, $m = 10$), *perfect* if $\sigma(m) = 2m$ (for example, $m = 6$), and *abundant* if $\sigma(m) > 2m$ (for example, $m = 12$). This classification goes back to the ancient Greeks; however, it was only in the 20th century that significant progress was made in understanding how these numbers were distributed within the sequence of natural numbers. For each $u \in [0,1]$ and each real $x \ge 1$, put

$$\mathscr{D}(u;x) := \left\{m \le x : \frac{m}{\sigma(m)} \le u\right\}, \quad \text{and put} \quad D(u;x) := \#\mathscr{D}(u;x).$$

2010 Mathematics Subject Classification: 11N60, 11A25.
Keywords: abundant number, deficient number, distribution function, Erdős–Wintner theorem.

In 1933, Davenport [4] showed that for all $u \in [0,1]$, the limit

$$D(u) := \lim_{x\to\infty} \frac{1}{x} D(u;x)$$

exists. Moreover, $D(u)$ is a continuous function of u, with $D(0) = 0$ and $D(1) = 1$. From these results, one quickly deduces that the deficient numbers have natural density $1 - D(\frac{1}{2})$, that the perfect numbers have density 0 (here one uses the continuity of $D(u)$), and that the abundant numbers have density $D(\frac{1}{2})$. It is of some interest to obtain accurate numerical approximations of these values; improving on much earlier work, Kobayashi [10] has recently shown that the density of the abundant numbers lies between 0.24761 and 0.24766.

In 1946, Erdős [5] showed that the abundant and deficient numbers have the distribution predicted by Davenport's result even in remarkably short intervals (see [5, Theorem 7(iii)]; see also [1] for closely related material). In fact, he showed that if $A = A(x) \to \infty$, then

$$\lim_{x\to\infty} \frac{\#\{m \in (x, x + A\log_3 x] : \frac{m}{\sigma(m)} \le u\}}{A \log_3 x} = D(u)$$

for every fixed $u \in [0,1]$. The primary purpose of this note is to illustrate how Erdős's ideas may be adapted to study the distribution of abundant and deficient numbers in arithmetic progressions. Specifically, we establish a sufficient condition for the elements of $\mathscr{D}(u;x)$ to approach equidistribution modulo q, as both q and x tend to infinity. Our proof uses the same ideas that feature in Erdős's work [5], supplemented by the method of moments.

It is certainly necessary to assume that $x \to \infty$ to meaningfully discuss equidistribution, but why assume that $q \to \infty$? It turns out that for fixed $q > 1$ and $u \in (0,1]$, the elements of $\mathscr{D}(u;x)$ do not approach equidistribution as $x \to \infty$. Let us quickly explain why. Consider those $m \in \mathscr{D}(u;x)$ which are 0 mod q. Included here are all $m \in [1,x]$ of the form nq, where $n/\sigma(n) \le u$. The density of n satifying $n/\sigma(n) \le u$ is $D(u)$, which already implies that the limiting proportion of elements of $\mathscr{D}(u;x)$ that are 0 mod q is at least $1/q$. However, there are many m still unaccounted for! For instance, a positive proportion of natural numbers n are both coprime to q and satisfy $u < n/\sigma(n) \le u\sigma(q)/q$. (The proof of this parallels the proof that Davenport's distribution function D is strictly increasing; compare with [13, Exercise 35, p. 275].) For these n, the number $m = nq$ also satisfies $m/\sigma(m) \le u$. It follows that the lower density of $m \equiv 0 \pmod q$ satisfying $m/\sigma(m) \le u$ is *strictly larger* than $\frac{1}{q}D(u)$, contradicting equidistribution. This analysis can easily be extended beyond the residue class 0 mod q to all of the residue classes a mod q with $\gcd(a,q) > 1$.

Thus, equidistribution for fixed moduli q is not in the cards. So to obtain equidistribution results, we must allow q to vary with x. To avoid the difficulties discussed in the last paragraph, we also assume that $\sigma(q)/q = 1 + o(1)$.

Definition. Let $\mathscr{Q}$ be an infinite set of natural numbers. We say that $\mathscr{Q}$ is *asymptotically tame* if $\frac{\sigma(q)}{q} \to 1$ as $q \to \infty$ through elements of $\mathscr{Q}$.

For instance, the prime numbers form an example of an asymptotically tame set. Our main theorem establishes equidistribution in a wide range of q and x provided that q is restricted to an asymptotically tame set. We write $\log_k x$ for the kth iterate of the function $\log_1 x := \max\{1, \log x\}$.

Theorem 1.1. *Let $\mathscr{Q}$ be an asymptotically tame set of natural numbers. Let u be a fixed real number with $0 < u \leq 1$. For q restricted to $\mathscr{Q}$, we have that whenever q, x, and $\frac{x}{q \log_3 x}$ all tend to infinity,*

$$\frac{\#\{m \leq x : m \equiv a \bmod q,\, \frac{m}{\sigma(m)} \leq u\}}{\#\{m \leq x : m \equiv a \bmod q\}} \to D(u),$$

uniformly in the choice of residue class $a \bmod q$.

To see that this really is an equidistribution result, note that the denominator here is $\sim x/q$, while $D(u;x) \sim D(u)x$; thus, the proportion of elements of $\mathscr{D}(u;x)$ belonging to the progression $a \bmod q$ is asymptotically $1/q$. As we explain after the proof of this theorem, the range of uniformity in q is in some sense sharp.

The method of moments can also be used to establish several closely related results. Rather than try to formulate the most general theorem possible, we focus on a single theorem that is fairly representative of what may be expected.

Recall that every prime p possesses $\varphi(p-1)$ primitive roots (i.e., generators of the multiplicative group modulo p). Work of Burgess [3] shows that for $X := p^{\frac{1}{4}+\epsilon}$, the number of primitive roots mod p in $[1, X]$ is asymptotic to $\frac{\varphi(p-1)}{p-1}X$ as $p \to \infty$. Our second theorem shows that these small primitive roots also follow Davenport's distribution.

Theorem 1.2. *Fix $\epsilon > 0$ and fix $u \in (0,1]$. As p tends to infinity through prime values,*

$$\frac{\#\{\text{primitive roots } 1 \leq m \leq p^{\frac{1}{4}+\epsilon} : \frac{m}{\sigma(m)} \leq u\}}{\#\{\text{primitive roots } 1 \leq m \leq p^{\frac{1}{4}+\epsilon}\}} \to D(u).$$

Notation and conventions

Throughout, we reserve the letter p for a prime variable. We employ O and o-notation, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with the usual meanings. We write $p^e \parallel m$ to mean that $p^e \mid m$ but that $p^{e+1} \nmid m$. We use $\omega(m)$ for the number of distinct primes dividing m. Other notation will be introduced as necessary.

2. Preliminary remarks on the moments of $\frac{n}{\sigma(n)}$

The proofs of both theorems hinge on the following well-known result. See, for example, the textbook of Billingsley [2, Theorems 30.1 and 30.2, pp. 406–408].

Lemma 2.1. *Let $F_1, F_2, F_3, \ldots$ be a sequence of distribution functions. Suppose that each F_n corresponds to a probability measure on the real line concentrated on $[0,1]$. For each $k = 1, 2, 3, \ldots,$ assume that*

$$\mu_k := \lim_{n\to\infty} \int u^k \, \mathrm{d}F_n(u)$$

exists. Then there is a unique distribution function F possessing the μ_k as its moments, and F_n converges weakly to F.

In order to apply Lemma 2.1, we will need a convenient expression for the moments of Davenport's distribution function D.

Lemma 2.2. *Let k be a natural number. The kth moment of $D(u)$ is given by the absolutely convergent sum*

$$\mu_k = \sum_{d_1,\ldots,d_k} \frac{g(d_1)\cdots g(d_k)}{\operatorname{lcm}[d_1,\ldots,d_k]}. \tag{2.1}$$

Here the d_i run over all natural numbers, and the multiplicative function g is defined by the convolution identity

$$\frac{n}{\sigma(n)} = \sum_{d\mid n} g(d).$$

Proof. For each natural number M, let $D_M(u) := \frac{1}{M}\#\{m \le M : \frac{m}{\sigma(m)} \le u\}$. Then the distribution functions D_M converge weakly to D; as a consequence,

$$\int u^k \, \mathrm{d}D(u) = \lim_{M\to\infty} \int u^k \, \mathrm{d}D_M(u).$$

(Compare with [2, Corollary, p. 348].) We now calculate

$$\int u^k \, \mathrm{d}D_M(u) = \frac{1}{M} \sum_{m\le M} (m/\sigma(m))^k$$

$$= \frac{1}{M} \sum_{d_1,\dots,d_k \le M} g(d_1)\cdots g(d_k) \sum_{\substack{m \le M \\ \mathrm{lcm}[d_1,\dots,d_k] \mid m}} 1.$$

The final sum is $M/\mathrm{lcm}[d_1,\dots,d_k] + O(1)$, which shows that the kth moment of D_M is given by

$$\sum_{d_1,\dots,d_k \le M} \frac{g(d_1)\cdots g(d_k)}{\mathrm{lcm}[d_1,\dots,d_k]} + O\left(\frac{1}{M} \sum_{d_1,\dots,d_k \le M} |g(d_1)\cdots g(d_k)|\right). \tag{2.2}$$

On each prime power p^e, we find that

$$g(p^e) = \frac{p^e}{\sigma(p^e)} - \frac{p^{e-1}}{\sigma(p^{e-1})} = -\frac{p^{e-1}}{\sigma(p^e)\sigma(p^{e-1})},$$

and so in particular, $|g(p^e)| < 1/p^e$. Consequently, $|g(n)| \le 1/n$ for every n. Hence, the error term in (2.2) is $O(\frac{1}{M}(1+\log M)^k)$, which vanishes as $M \to \infty$. If we show that the sum (2.1) defining μ_k is absolutely convergent, then taking the limit as $M \to \infty$ in (2.2) will complete the proof of the lemma. But absolute convergence follows immediately from the bounds

$$\left|\frac{g(d_1)\cdots g(d_k)}{\mathrm{lcm}[d_1,\dots,d_k]}\right| \le \frac{1}{d_1\cdots d_k \cdot \mathrm{lcm}[d_1,\dots,d_k]} \le \prod_{i=1}^{k} \frac{1}{d_i^{1+1/k}},$$

using for the final inequality that

$$\mathrm{lcm}[d_1,\dots,d_k] \ge \max\{d_1,\dots,d_k\} \ge (d_1\cdots d_k)^{1/k}. \qquad \square$$

3. Proof of Theorem 1.1

The theorem is equivalent to the following proposition asserting weak convergence of certain distribution functions. Let $\mathscr{Q}$ be an asymptotically tame set, and let $\{x_j\}$, $\{q_j\}$, and $\{a_j\}$ be sequences satisfying the following three conditions:

(i) each $x_j \ge 1$, and $x_j \to \infty$ as $j \to \infty$,

(ii) each $q_j \in \mathscr{Q}$, and both q_j and $\frac{x_j}{q_j \log_3 x_j}$ tend to infinity,

(iii) each $a_j \in \mathbf{Z}$.

If these conditions are satisfied, we write $A_j := \frac{x_j}{q_j \log_3 x_j}$, so that $A_j \to \infty$ as $j \to \infty$.

For each j, let D_j be the distribution function defined by

$$D_j(u) := \frac{\#\{m \le x_j : m \equiv a_j \bmod q_j \text{ and } \frac{m}{\sigma(m)} \le u\}}{\#\{m \le x_j : m \equiv a_j \bmod q_j\}}. \tag{3.1}$$

Proposition 3.1. *As $j \to \infty$, D_j converges weakly to D.*

From Lemma 2.1, the proposition will follow if we can show that for each fixed k,

$$\lim_{j\to\infty} \int u^k \,\mathrm{d}D_j(u) = \mu_k,$$

with the μ_k as defined in (2.1). To begin with, we compute that

$$\int u^k \,\mathrm{d}D_j(u) = \frac{\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} (m/\sigma(m))^k}{\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} 1}. \tag{3.2}$$

As $j\to\infty$, the denominator in (3.2) is asymptotic to x_j/q_j. We turn now to a study of the numerator. In what follows, we adopt the notation

$$m' := \prod_{\substack{p^e\| m\\ p\mid q_j}} p^e, \quad m'' := \prod_{\substack{p^e\| m\\ p\nmid q_j,\ p\le \log_3 x_j}} p^e, \quad m''' := \prod_{\substack{p^e\| m\\ p\nmid q_j,\ p>\log_3 x_j}} p^e;$$

clearly,

$$m = m'm''m'''.$$

First we work on an upper bound. Since $\frac{m}{\sigma(m)} \le \frac{m''}{\sigma(m'')}$, we have

$$\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} \left(\frac{m}{\sigma(m)}\right)^k \le \sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} \left(\frac{m''}{\sigma(m'')}\right)^k.$$

Recalling the definition of the arithmetic function g, we see that

$$\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} \left(\frac{m''}{\sigma(m'')}\right)^k = \sum_{\substack{d_1,\dots,d_k\le x_j\\ p\mid d_i \Rightarrow p\le \log_3 x_j\\ \gcd(d_i,q)=1}} g(d_1)\cdots g(d_k) \sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j\\ \mathrm{lcm}[d_1,\dots,d_k]\mid m}} 1.$$

The inner sum on the right-hand side is $\frac{x_j}{q_j \,\mathrm{lcm}[d_1,\dots,d_k]} + O(1)$, which shows that this last expression is

$$\frac{x_j}{q_j} \sum_{\substack{d_1,\dots,d_k\le x_j\\ p\mid d_i \Rightarrow p\le \log_3 x_j\\ \gcd(d_1\cdots d_k,q_j)=1}} \frac{g(d_1)\cdots g(d_k)}{\mathrm{lcm}[d_1,\dots,d_k]} + O\Bigg(\sum_{\substack{d_1,\dots,d_k\le x_j\\ p\mid d_i \Rightarrow p\le \log_3 x_j\\ \gcd(d_1\cdots d_k,q_j)=1}} |g(d_1)\cdots g(d_k)|\Bigg). \tag{3.3}$$

To estimate the error, we recall that $|g(d)| \le 1/d$, so that

$$\Bigg|\sum_{\substack{d_1,\dots,d_k\le x_j\\ p\mid d_i\Rightarrow p\le \log_3 x_j\\ \gcd(d_1\cdots d_k,q)=1}} g(d_1)\cdots g(d_k)\Bigg| \le \Bigg(\sum_{\substack{d\ge 1\\ p\mid d\Rightarrow p\le\log_3 x_j}} \frac{1}{d}\Bigg)^k$$
$$= \prod_{p\le \log_3 x_j}\left(1-\frac{1}{p}\right)^{-k} \le (2\log_4 x_j)^k$$

once j is large. (We use Mertens' theorem here as well as the bound $e^\gamma < 2$.) Since $x_j/q_j = A_j \log_3 x_j$, where $A_j\to\infty$, we see that the error term in (3.3) is $o(x_j/q_j)$. Thus,

$$\frac{1}{x_j/q_j}\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}}\left(\frac{m''}{\sigma(m'')}\right)^k = \sum_{\substack{d_1,\dots,d_k\le x_j\\ p\mid d_i\Rightarrow p\le\log_3 x_j\\ \gcd(d_1\cdots d_k,q_j)=1}} \frac{g(d_1)\cdots g(d_k)}{\operatorname{lcm}[d_1,\dots,d_k]} + o(1), \qquad (3.4)$$

as $j\to\infty$. Referring back to (3.2) and remembering that the sum defining μ_k in (2.1) converges absolutely, we deduce that

$$\limsup_{j\to\infty}\int u^k\,\mathrm{d}D_j(u) \le \limsup_{j\to\infty}\sum_{\substack{d_1,\dots,d_k\\ \gcd(d_1,\dots,d_k,q_j)=1}}\frac{g(d_1)\cdots g(d_k)}{\operatorname{lcm}[d_1,\dots,d_k]}. \qquad (3.5)$$

Next, we develop an analogous lower bound. Fix a small positive ϵ, say $\epsilon\in(0,\frac{1}{2})$. We claim that the number of $m\le x_j$ in the progression $a_j \bmod q_j$ for which

$$\frac{m'''}{\sigma(m''')} < 1-\epsilon$$

is $o(x_j/q_j)$, as $j\to\infty$. This claim will be deduced from an upper bound on the product of the terms $\frac{\sigma(m''')}{m'''}$. For each prime power p^e, we have $\frac{\sigma(p^e)}{p^e} < \frac{p}{p-1}$. Consequently,

$$\prod_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}}\frac{\sigma(m''')}{m'''} \le \prod_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}}\ \prod_{\substack{p\mid m\\ p\nmid q\\ p>\log_3 x_j}}\frac{p}{p-1}$$
$$= \exp\Bigg(\sum_{\substack{p\le x_j\\ p\nmid q,\ p>\log_3 x_j}}\log\frac{p}{p-1}\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j\\ p\mid m}} 1\Bigg). \qquad (3.6)$$

Note that

$$\log\frac{p}{p-1} = \log\left(1+\frac{1}{p-1}\right) < \frac{1}{p-1} \le \frac{2}{p}.$$

For primes $p \le x_j/q_j$, the inner sum in (3.6) is at most $1 + \frac{x_j}{q_j p} \le 2\frac{x_j}{q_j p}$, and so the contribution to the double sum from these primes is at most

$$2\frac{x_j}{q_j}\sum_{p>\log_3 x_j}\frac{1}{p}\log\frac{p}{p-1} \le 4\frac{x_j}{q_j}\sum_{p>\log_3 x_j}\frac{1}{p^2} < \frac{x_j}{q_j\log_3 x_j},$$

once j is large (using partial summation and the prime number theorem in the final step). Now suppose that p is a prime not dividing q_j with $p > x_j/q_j$. Then p can divide at most one integer $m \le x_j$ from the progression $a_j \bmod q_j$, since the difference between any two such m has the form $q_j\ell$ with $\ell \le x_j/q_j$. So letting

$$\Pi := \prod_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} m,$$

we see that

$$\sum_{\substack{p>x_j/q_j\\ p\nmid q,\ p>\log_3 x_j}}\log\frac{p}{p-1}\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j\\ p\mid m}} 1 \le 2\sum_{p\mid\Pi}\frac{1}{p}.$$

Now $\Pi \le (x_j)^{1+x_j/q_j} \le x_j^{2x_j/q_j}$, and so by the prime number theorem,

$$\Pi \le \prod_{p\le 4\frac{x_j}{q_j}\log(x_j)} p.$$

(We assume here, as we may, that j is large.) Thus, $\omega(\Pi)$ is at most the total count of primes up to $4\frac{x_j}{q_j}\log(x_j)$, and the sum of $\frac{1}{p}$ taken over the primes dividing Π is bounded above by the corresponding sum over the primes up to $4\frac{x_j}{q_j}\log(x_j)$. As a consequence,

$$\begin{aligned}2\sum_{p\mid\Pi}\frac{1}{p} \le 2\sum_{p\le 4\frac{x_j}{q_j}\log(x_j)}\frac{1}{p} &\le 2\log\log(4\frac{x_j}{q_j}\log(x_j)) + O(1)\\ &\le 2\log_2(x_j/q_j) + 2\log_3 x_j + O(1).\end{aligned}$$

Collecting our estimates and referring back to (3.6), we find that

$$\prod_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}}\frac{\sigma(m''')}{m'''} \ll \exp\left(\frac{x_j}{q_j\log_3 x_j}\right)(\log(x_j/q_j))^2(\log_2 x_j)^2.$$

On the other hand, whenever $\frac{m'''}{\sigma(m''')} < 1-\epsilon$, we have $\frac{\sigma(m''')}{m'''} > 1+\epsilon$; hence, the number of these $m \le x$ is at most

$$\frac{\log \prod_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} \frac{\sigma(m''')}{m'''}}{\log(1+\epsilon)} \ll_\epsilon 1 + \frac{x_j}{q_j \log_3 x_j} + \log_2(x_j/q_j) + \log_3 x_j.$$

The first three terms on the right-hand side are clearly $o(x_j/q_j)$ as $j \to \infty$. The last one is also, since $x_j/q_j = A_j \log_3 x_j$, where $A_j \to \infty$. This proves the claim.

Now recalling the tameness assumption and the identity $\frac{\sigma(q_j)}{q_j} = \sum_{d\mid q_j} \frac{1}{d}$, we get that

$$\frac{m'}{\sigma(m')} \ge \prod_{p\mid q_j}\left(1-\frac{1}{p}\right) \ge 1 - \sum_{p\mid q_j}\frac{1}{p} \ge 1 - \left(\frac{\sigma(q_j)}{q_j} - 1\right) = 1 - o(1)$$

as $j \to \infty$, uniformly in m. In particular, once j is large, we always have

$$\frac{m'}{\sigma(m')} \ge 1-\epsilon.$$

Using $*$ for a sum restricted to m having $\frac{m'''}{\sigma(m''')} > 1-\epsilon$, we deduce that for large j,

$$\begin{aligned}\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} \left(\frac{m}{\sigma(m)}\right)^k &\ge \sideset{}{^*}\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} \left(\frac{m'}{\sigma(m')}\right)^k \left(\frac{m''}{\sigma(m'')}\right)^k \left(\frac{m'''}{\sigma(m''')}\right)^k\\ &\ge (1-\epsilon)^{2k} \sideset{}{^*}\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} \left(\frac{m''}{\sigma(m'')}\right)^k.\end{aligned}$$

The final restricted sum differs from the corresponding unrestricted sum by at most $o(x_j/q_j)$, since the restriction only removes $o(x_j/q_j)$ terms (by our earlier claim), each of which is nonnegative and at most 1. Thus,

$$\begin{aligned}\liminf_{j\to\infty} \int u^k \,\mathrm{d}D_j(u) &= \liminf_{j\to\infty} \frac{\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} (m/\sigma(m))^k}{x_j/q_j}\\ &\ge (1-\epsilon)^{2k} \cdot \liminf_{j\to\infty} \frac{\sum_{\substack{m\le x_j\\ m\equiv a_j \bmod q_j}} (m''/\sigma(m''))^k}{x_j/q_j}.\end{aligned}$$

Since ϵ may be taken arbitrarily small, the last inequality remains valid without the factor of $(1-\epsilon)^{2k}$. Now referring back to (3.4), we find that

$$\liminf_{j\to\infty} \int u^k \,\mathrm{d}D_j(u) \geq \liminf_{j\to\infty} \sum_{\substack{d_1,\ldots,d_k \\ \gcd(d_1\cdots d_k, q_j)=1}} \frac{g(d_1)\cdots g(d_k)}{\operatorname{lcm}[d_1,\ldots,d_k]}. \tag{3.7}$$

Comparing (3.5) and (3.7), we see that our proposition will be proved if we show that

$$\lim_{j\to\infty} \sum_{\substack{d_1,\ldots,d_k \\ \gcd(d_1\cdots d_k, q_j)=1}} \frac{g(d_1)\cdots g(d_k)}{\operatorname{lcm}[d_1,\ldots,d_k]} = \sum_{d_1,\ldots,d_k} \frac{g(d_1)\cdots g(d_k)}{\operatorname{lcm}[d_1,\ldots,d_k]}.$$

Using once again that $|g(d)| \leq 1/d$ for all d, we see that

$$\left| \sum_{d_1,\ldots,d_k} \frac{g(d_1)\cdots g(d_k)}{\operatorname{lcm}[d_1,\ldots,d_k]} - \sum_{\substack{d_1,\ldots,d_k \\ \gcd(d_1\cdots d_k, q_j)=1}} \frac{g(d_1)\cdots g(d_k)}{\operatorname{lcm}[d_1,\ldots,d_k]} \right| \\ \leq \sum_{p \mid q_j} \sum_{i=1}^{k} \sum_{\substack{d_1,\ldots,d_k \\ p \mid d_i}} \frac{1}{d_1\cdots d_k \cdot \operatorname{lcm}[d_1,\ldots,d_k]}.$$

Writing $d_i = pd_i'$, the inner summand is at most

$$\frac{1}{pd_1\cdots d_{i-1}d_i'd_{i+1}\cdots d_k \operatorname{lcm}[d_1,\cdots,d_{i-1},d_i',d_{i+1},\cdots,d_k]},$$

and so the triple sum is crudely bounded above by

$$k\left(\sum_{p\mid q_j}\frac{1}{p}\right)\sum_{d_1,\ldots,d_k}\frac{1}{d_1\cdots d_k\cdot\operatorname{lcm}[d_1,\ldots,d_k]} \\ \leq k\left(\frac{\sigma(q_j)}{q_j}-1\right)\sum_{d_1,\ldots,d_k}\frac{1}{(d_1\cdots d_k)^{1+1/k}} = k\left(\frac{\sigma(q_j)}{q_j}-1\right)\zeta(1+1/k)^k.$$

But as $j\to\infty$, the final expression tends to 0 by the tameness hypothesis. This completes the proof.

Optimality

We might wonder whether, instead of assuming that $\frac{x}{q\log_3 x}\to\infty$, we can get by with the weaker assumption that $\frac{x}{q\log_3 x}$ is sufficiently large. Equivalently, we might wonder whether there is a large absolute constant A so that Proposition 3.1 is true with condition (ii) replaced by (ii′) each $q_j \in \mathscr{Q}$, $q_j \to \infty$, and each $q_j \le \frac{x_j}{A \log_3 x_j}$.

But this is not so. To see this, it is enough to show that no matter how large A is taken, there is an asymptotically tame set $\mathscr{Q}$ and sequences $\{x_j\}, \{q_j\}$, and $\{a_j\}$ satisfying conditions (i), (ii′), and (iii) for which the corresponding distribution functions D_j, as defined in (3.1), do not converge weakly to D. In fact, we will show that these conditions do not even guarantee that the first moments of D_j approach the first moment of D. The argument is closely analogous to one presented by Erdős in detail (see [5, p. 532]), and so we only outline it.

- We let $\mathscr{Q}$ be the set of natural numbers q whose smallest prime factor exceeds $\frac{1}{10}\log q$. Each $q \in \mathscr{Q}$ satisfies

$$1 \le \frac{\sigma(q)}{q} \le \exp\left(\sum_{p \mid q} \frac{1}{p-1}\right) \le \exp\left(\frac{20\omega(q)}{\log q}\right).$$

  Since the maximal order of $\omega(q)$ is $\log q/\log\log q$ (cf. [7, p. 471]), the set $\mathscr{Q}$ is asymptotically tame.

- Let $\{x_j\}$ be a sequence that tends to infinity. We will also assume at various points that all of the x_j are sufficiently large (possibly depending on A). For each j, let $t = t_j = \lfloor \log_3 x_j \rfloor$, and choose squarefree numbers $n_{j,1}, n_{j,2}, \ldots, n_{j,t-1}$, all supported on disjoint subsets of the primes in $(\log_3 x_j, \frac{1}{10}\log x_j]$ and satisfying

$$\frac{n_{j,i}}{\sigma(n_{j,i})} \le e^{-9/10}.$$

  Since $\sum_{\log_3 x_j < p \le \frac{1}{10}\log x_j} \log \frac{p}{\sigma(p)} \sim -\log_3 x_j$, this is easily seen to be possible by employing a greedy construction.

- Choose q_j with

$$\frac{x_j}{2A\log_3 x_j} < q_j < \frac{x_j}{A\log_3 x_j} \tag{3.8}$$

  as a solution to the system of simultaneous congruences

$$q_j \equiv 1 \bmod \prod_{\substack{p \le \frac{1}{10}\log x_j \\ p \nmid n_{j,1} n_{j,2} \cdots n_{j,t-1}}} p,$$
$$\text{and} \quad iq_j + 1 \equiv 0 \bmod n_{j,i} \quad (\text{for } 1 \le i < t_j).$$

  (Note that $i < \log_3 x_j$, so that i is invertible modulo $n_{j,i}$.) The Chinese remainder theorem lets us to do this, since $\prod_{p \le \frac{1}{10}\log x_j} p \le x_j^{1/5} < x_j/(2A\log_3 x_j)$.

  Each q_j has smallest prime factor $> \frac{1}{10}\log x_j > \frac{1}{10}\log q_j$, and so $q_j \in \mathscr{Q}$.

- Finally, we choose each $a_j = 1$. This finishes the selection of the set $\mathscr{Q}$ and the sequences $\{x_j\}, \{q_j\}$, and $\{a_j\}$.
- Despite (i), (ii′), and (iii) all being satisfied, the first moments of the D_j turn out to be too small. To make this precise, let Δ be the first moment of D, so that

$$\Delta = \prod_p \left(\sum_{e=1}^{\infty} g(p^e)/p^e \right) \approx 0.67.$$

Then we can show that

$$\limsup_{j\to\infty} \frac{1}{x_j/q_j} \sum_{\substack{m\le x_j\\ m\equiv 1 \bmod q_j}} \frac{m}{\sigma(m)} < \Delta. \tag{3.9}$$

Note that the left-hand side here is the lim sup of the first moments of the D_j.

To see why (3.9) holds, observe that we have rigged the behavior of the first several terms of the sum. Indeed, all of the terms $1 < m < t_j q_j$ that appear have the form $m = iq_j + 1$ for some $1 \le i < t_j$, and so $m/\sigma(m) \le e^{-9/10}$. Thus, these terms contribute at most $e^{-9/10} t_j$ to the sum. We bound the contribution of the terms $m > t_j q_j$ by replacing $m/\sigma(m)$ with $m''/\sigma(m'')$ and mimicking the upper bound argument of the theorem. (Note that it was not important there that $A \to \infty$.) We find that the remaining terms make a contribution to the sum of size at most $(x_j/q_j - t_j)\Delta + o(x_j/q_j)$. Piecing everything together, we find that the left-hand side of (3.9) is at most

$$\limsup_{j\to\infty} \left(e^{-9/10} \frac{t_j}{x_j/q_j} + \Delta\left(1 - \frac{t_j}{x_j/q_j}\right)\right).$$

The expression inside the lim sup is a weighted average (convex combination) of $e^{-9/10} \approx 0.41$ and $\Delta \approx 0.67$; moreover, the coefficient of $e^{-9/10}$ in this convex combination is $\gg_A 1$, because of (3.8). This is enough to guarantee that the rigged terms skew the lim sup in (3.9) below Δ.

4. Proof of Theorem 1.2

We begin by quoting the following version of Burgess's character sum estimate, a proof of which can be found in the text of Iwaniec and Kowalski [9, pp. 327–329].

Lemma 4.1. *Let p be a prime, and let χ be a nontrivial Dirichlet character mod p. Let M and N be integers with $N > 0$, and let r be a positive integer. Then*

$$\sum_{M<n\leq M+N} \chi(n) \ll N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}.$$

Here the implied constant is absolute.

We will also need the following result expressing the characteristic function of the primitive roots modulo p in terms of Dirichlet characters (see [3, Lemma 5]).

Lemma 4.2. *Let p be a prime number. For each integer m, let*

$$\xi(m) = \frac{\varphi(p-1)}{p-1}\left(\chi_0(m) + \sum_{\substack{d\mid p-1\\ d>1}} \frac{\mu(d)}{\varphi(d)} \sum_{\chi \text{ of order } d} \chi(m)\right),$$

where χ_0 denotes the principal character mod p and the sum on χ is over those characters of exact order d. Then $\xi(m) = 1$ if m is a primitive root mod p, and $\xi(m) = 0$ otherwise.

We can now commence the proof of Theorem 1.2. For each prime p, let $X = p^{\frac{1}{4}+\epsilon}$ and introduce the distribution function

$$D_p(u) := \frac{\#\{\text{primitive roots } 1 \leq m \leq X : \frac{m}{\sigma(m)} \leq u\}}{\#\{\text{primitive roots } 1 \leq m \leq X\}}.$$

We will show that for each fixed positive integer k, the kth moment of D_p converges to the kth moment μ_k of Davenport's distribution function D, as $p \to \infty$. We start by writing

$$\int u^k \,\mathrm{d}D_p(u) = \frac{1}{\#\{\text{primitive roots } 1 \leq m \leq X\}} \sum_{\substack{m\leq X\\ m \text{ primitive root}}} \left(\frac{m}{\sigma(m)}\right)^k. \tag{4.1}$$

In [3], it is shown that the count of primitive roots in $[1, X]$ is asymptotic to $\frac{\varphi(p-1)}{p-1}X$ as $p \to \infty$, and so we focus our attention on the estimation of the sum in (4.1). Using ξ for the function defined in Lemma 4.2,

$$\sum_{\substack{m\leq X\\ m \text{ primitive root}}} \left(\frac{m}{\sigma(m)}\right)^k = \sum_{m\leq X} \xi(m) \left(\frac{m}{\sigma(m)}\right)^k,$$

which can be expanded as

$$\frac{\varphi(p-1)}{p-1}\left(\sum_{\substack{m\le X\\ p\nmid m}}\left(\frac{m}{\sigma(m)}\right)^k+\sum_{\substack{d\mid p-1\\ d>1}}\frac{\mu(d)}{\varphi(d)}\sum_{\chi\text{ of order }d}\sum_{m\le X}\chi(m)\left(\frac{m}{\sigma(m)}\right)^k\right). \tag{4.2}$$

Applying Lemma 4.1, with r a parameter to be chosen momentarily, we get

$$\begin{aligned}\sum_{m\le X}\chi(m)\left(\frac{m}{\sigma(m)}\right)^k &= \sum_{m\le X}\chi(m)\left(\sum_{d\mid m}g(d)\right)^k\\ &= \sum_{d_1,\dots,d_k\le X}g(d_1)\cdots g(d_k)\chi(\mathrm{lcm}[d_1,\dots,d_k])\sum_{n\le\frac{X}{\mathrm{lcm}[d_1,\dots,d_k]}}\chi(n)\\ &\ll X^{1-\frac{1}{r}}p^{\frac{r+1}{4r^2}}(\log p)^{1/r}\sum_{d_1,\dots,d_k\le X}\frac{|g(d_1)|\cdots|g(d_k)|}{\mathrm{lcm}[d_1,\dots,d_k]^{1-\frac{1}{r}}}.\end{aligned}$$

We now assume that $r\ge 2$. Using that each $|g(d_i)|\le 1/d_i$ and that

$$\mathrm{lcm}[d_1,\dots,d_k]\ge(d_1\dots d_k)^{1/k},$$

we find that the remaining sum on the d_i is $O_k(1)$. Since there are precisely $\varphi(d)$ characters χ of order d, we deduce that

$$\begin{aligned}\sum_{\substack{d\mid p-1\\ d>1}}\frac{\mu(d)}{\varphi(d)}\sum_{\chi\text{ of order }d}\sum_{m\le X}\chi(m)\left(\frac{m}{\sigma(m)}\right)^k &\ll_k X^{1-\frac{1}{r}}p^{\frac{r+1}{4r^2}}(\log p)^{1/r}\sum_{d\mid p-1}|\mu(d)|\\ &= (2^{\omega(p-1)}(\log p)^{1/r}p^{-\frac{\epsilon}{r}+\frac{1}{4r^2}})X.\end{aligned}$$

Now choosing $r:=\max\{2,1+\lfloor(4\epsilon)^{-1}\rfloor\}$, we obtain after a quick computation that this last expression is $O_\epsilon(X^{1-\delta})$ for a certain $\delta=\delta(\epsilon)>0$.

Moreover, $\sum_{m\le X}(m/\sigma(m))^k\sim\mu_k X$ as $p\to\infty$. Removing the terms in this sum with m divisible by p (which appear only when $\epsilon\ge\frac{3}{4}$) changes the sum by $O(X/p)$, which is $o(X)$ as $p\to\infty$. Hence,

$$\sum_{\substack{m\le X\\ p\nmid m}}\left(\frac{m}{\sigma(m)}\right)^k\sim\mu_k X.$$

Piecing things together, we conclude that the initial sum in (4.2) is asymptotic to $\frac{\varphi(p-1)}{p-1}\mu_k X$ as $X\to\infty$. Combining this with our earlier estimate for the denominator in (4.1), we see that the kth moment of D_p tends to μ_k as $p \to \infty$, as desired.

5. Concluding remarks

Narkiewicz has called a set of integers *weakly equidistributed modulo q* if the elements of the set that are coprime to q are uniformly distributed among the coprime residue classes modulo q. (See, for example, [12].) Suppose that $u \in (0,1]$ is fixed. Then the elements of $\mathscr{D}(u;x)$ become weakly equidistributed modulo each fixed q, as $x \to \infty$. More precisely, for every fixed coprime residue class $a \bmod q$,

$$\#\{m \le x : m \equiv a \bmod q, \frac{m}{\sigma(m)} \le u\} \sim \frac{1}{\varphi(q)} \#\{m \le x : \gcd(m,q) = 1, \frac{m}{\sigma(m)} \le u\}, \quad (5.1)$$

as $x \to \infty$. The weaker verison of this claim, where logarithmic density takes the place of natural density, is a consequence of [6, Lemma 1.17, p. 61]. A full proof of (5.1) can be obtained either through the method of moments or by the more concrete methods of [11]. In fact, the moments argument is used in [14, Lemma 2.2] to show that the limiting proportion of m with $m/\sigma(m) \le u$ from a fixed residue class $a \bmod q$ is the same for all classes $a \bmod q$ sharing the same value of $\gcd(a,q)$.

Somewhat frustratingly, none of the methods alluded to in the last paragraph seem well-suited to establishing an analogue of Theorem 1.1, i.e., showing that for fixed $u > 1$, the asymptotic relation (5.1) holds uniformly in a wide range of q. Some care will be necessary to formulate the right conjecture here; Iannucci [8] has shown that if q is the product of the primes up to $(\log x)^{1/2+\epsilon}$, so that $q \approx \exp((\log x)^{1/2+\epsilon})$, then the interval $[1,x]$ contains no abundant numbers relatively prime to q.

Acknowledgements

The author thanks Greg Martin for useful comments, in particular for pointing out that the method of moments yields the asymptotic weak equidistribution of $\mathscr{D}(u;x)$, as $x \to \infty$, in the case of a fixed modulus q. He also thanks the referee for a careful reading of the manuscript.

REFERENCES

[1] BALÁŽ, V. – LIARDET, P. – STRAUCH, O.: *Distribution functions of the sequence $\phi(M)/M$, $M \in (k, k+N]$ as k, N go to infinity*, Integers **10** (2010), A53, 705–732.

[2] BILLINGSLEY, P.: *Probability and measure*, second ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986.

[3] BURGESS, D. A.: *On character sums and primitive roots*, Proc. London Math. Soc. (3) **12** (1962), 179–192.

[4] DAVENPORT, H.: *Über numeri abundantes*, S.-Ber. Preuß. Akad. Wiss., math.-nat. Kl. (1933), 830–837.

[5] ERDŐS, P.: *Some remarks about additive and multiplicative functions*, Bull. Amer. Math. Soc. **52** (1946), 527–537.

[6] HALL, R. R.: *Sets of multiples*, Cambridge Tracts in Mathematics, vol. **118**, Cambridge University Press, Cambridge, 1996.

[7] HARDY, G. H. – WRIGHT, E. M.: *An introduction to the theory of numbers*, sixth ed., Oxford University Press, Oxford, 2008.

[8] IANNUCCI, D. E.: *On the smallest abundant number not divisible by the first k primes*, Bull. Belg. Math. Soc. Simon Stevin **12** (2005), no. 1, 39–44.

[9] IWANIEC, H. – KOWALSKI, E.: *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. **53**, American Mathematical Society, Providence, RI, 2004.

[10] KOBAYASHI, M.: *On the density of abundant numbers*, Ph.D. thesis, Dartmouth College, 2010.

[11] KOBAYASHI, M. – POLLACK, P.: *The error term in the count of abundant numbers*, Mathematika, to appear.

[12] NARKIEWICZ, W.: *Uniform distribution of sequences of integers in residue classes*, Lecture Notes in Mathematics, vol. **1087**, Springer-Verlag, Berlin, 1984.

[13] POLLACK, P.: *Not always buried deep: A second course in elementary number theory*, American Mathematical Society, Providence, RI, 2009.

[14] POLLACK, P.: *Palindromic sums of proper divisors*, submitted.

Received June 1, 2011
Accepted June 1, 2011

Paul Pollack
Boyd Graduate Studies Building
Department of Mathematics
University of Georgia
Athens, Georgia 30602
USA
E-mail: pollack@uga.edu