ON RELATIVELY PRIME AMICABLE PAIRS

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Abstract. An *amicable pair* consists of two distinct numbers N and M each of which is the sum of the proper divisors of the other. For example, 220 and 284 form such a pair, as do 9773505 and 11791935. While over ten million such pairs are known, we know of no pair where N and M are relatively prime. Artjuhov and Borho have shown (independently) that if one fixes an upper bound on the number of distinct prime factors of NM, then there are only finitely many such coprime amicable pairs. We prove the following entirely explicit (but impractical) upper bound: If N and M form a coprime amicable pair with $\omega(NM) = K$, then

$$NM < (2K)^{2^{K^2}}.$$

1. Introduction

Distinct positive integers N and M are said to form an *amicable pair* if each is the sum of the proper divisors of the other; equivalently, $\sigma(N) = \sigma(M) = M + N$. Here $\sigma(\cdot)$ denotes the usual sum-of-divisors function. The smallest amicable pair, consisting of the numbers 220 and 284, was known already to the Pythagoreans (ca. 500 BCE). The extensive lore surrounding these numbers is entertainingly recounted in Dickson's *History* [Dic66, Chapter I], while a thorough account of modern developments can be found in the survey of Garcia, Pedersen, and te Riele [GPtR04].

In his 1917 dissertation [Gme17], Otto Gmelin noted that N and M share a nontrivial common factor in all the 62 then-known amicable pairs, and he conjectured somewhat cautiously that this was a universal phenomenon. Today, we know more than 10 million amicable pairs [Ped], none of which violate Gmelin's hypothesis. (In fairness, it must be admitted that many of these pairs are constructed by methods that could never produce a coprime pair.) While the empirical evidence is reasonably compelling, there seems to be no plausible strategy at present for proving Gmelin's conjecture.

A natural line of attack is to assume that there is such a pair and see what can be proved about it. From work of Hagis [Hag69, Hag70], we know that any such pair N, M has $NM > 10^{67}$. In fact, if N and M are assumed to have opposite parity, then 10^{67} can be replaced with 10^{121}. Hagis also showed [Hag72, Hag75] that

$$\omega(N) + \omega(M) \geq 22 \tag{1}$$

for any coprime amicable pair. Here $\omega(\cdot)$ is the arithmetic function which returns the number of distinct prime factors of its argument.

Borho [Bor74a, pp. 186–188] and Artjuhov [Art75] have each proven theorems implying that no matter what number is taken to replace 22, the number of coprime amicable pairs for which (1) fails is finite (see also the exposition [Pol12]). Neither author gives an explicit bound on the largest possible failure. Here we establish the following result.

Theorem 1. *Suppose that N and M form a relatively prime amicable pair with $\omega(NM) = K$. Then*

$$NM < (2K)^{2^{K^2}}.$$

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Remark. If we count prime factors *with multiplicity*, then the analogous problem is substantially simpler. Borho [Bor74b] has a one-page proof that an amicable pair N, M (*not* assumed coprime) with $\Omega(NM) = K$ has $NM < K^{2^K}$.

The theme taken up in this paper has a parallel in the theory of odd perfect numbers. Already in 1913, Dickson [Dic13] showed that for each fixed K, there are at most finitely many odd perfect N with $\omega(N) = K$. More than sixty years later, an explicit upper bound for N in terms of K was given by Pomerance [Pom77]. Pomerance's result was subsequently improved by Heath-Brown [HB94], Cook [Coo99], and Nielsen [Nie03]. Our proof of Theorem 1 is based on the method of Pomerance; it would be interesting to know if one can adapt the work of Heath-Brown and subsequent authors to obtain a sharper version of Theorem 1.

It is natural to ask whether one can prove a result in the same spirit as Theorem 1 where the coprimality condition on N and M is relaxed. The final section reports on some partial progress in this direction.

Notation. The letters p and q are reserved throughout for prime variables.

Suppose that n has the prime factorization $\prod_{i=1}^{k} p_i^{e_i}$, where $p_1 < p_2 < \dots < p_k$. We say d is a *prefix* of n if $d = \prod_{0 \le i \le j} p_i^{e_i}$ for some $j = 0, 1, 2, \dots, k$; we then write $d \mid^* n$. For example, $2 \mid^* 90$ and $18 \mid^* 90$, but $6 \nmid^* 90$ and $45 \nmid^* 90$. The *components* of n are the prime powers $p_1^{e_1}$, $p_2^{e_2}$, ..., $p_k^{e_k}$. We say that d is a *unitary divisor* of n if d is a product of some subset of the components of n; equivalently, $n = dd'$ with $\gcd(d, d') = 1$. We then write $d \parallel n$.

We write $h(n)$ for the multiplicative function $\sigma(n)/n$, so that on prime powers p^e, we have $h(p^e) = 1 + 1/p + \dots + 1/p^e$. We extend the domain of h to formal expressions of the form $\prod_p p^{e_p}$, where each $e_p \in \{0, 1, 2, \dots\} \cup \{\infty\}$ and $e_p = 0$ for all but finitely many p, by putting

$$h(\prod p^{e_p}) := \prod_p h(p^{e_p}), \quad \text{where} \quad h(p^\infty) := \lim_{e \to \infty} h(p^e) = \frac{p}{p-1}.$$

For example, $h(2 \cdot 3^\infty \cdot 5) = \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{6}{5}$. Note that for each prime p, the values $h(p^e)$ are strictly increasing for $0 \le e \le \infty$. In what follows, we make crucial use of the observation that for any amicable pair N and M,

$$\frac{1}{h(N)} + \frac{1}{h(M)} = \frac{N}{\sigma(N)} + \frac{M}{\sigma(M)} = \frac{N}{N+M} + \frac{M}{N+M} = 1.$$

2. Proof of Theorem 1

2.1. **An initial reduction.** We begin by reducing the proof of Theorem 1 to the problem of explicitly bounding above the largest prime factor of NM. This is accomplished by means of the following lemma, which is valid without any coprimality assumption.

Lemma 2. *Let K and B be integers with $K \ge 0$ and $B \ge 2$. Suppose that N and M form an amicable pair, that $\omega(N) + \omega(M) = K$, and that every prime dividing NM is bounded by B. Then*

$$NM \le (KB^K)^{2^K - 1}.$$

Remark. In particular, taking $K = 2\pi(B)$ gives an explicit upper bound on all amicable pairs for which NM has each of its prime factors bounded by B.

Proof. The lemma holds vacuously when $K = 0$, so we suppose that $K > 0$. We give an inductive procedure for successively discovering bounded components of N and M. Let $(N_0, M_0) := (N, M)$ and $(N_0', M_0') := (1, 1)$. Suppose $0 \le i < K$, that $N_i N_i' = N$, $M_i M_i' = M$, that $\gcd(N_i, N_i') = \gcd(M_i, M_i') = 1$, and that $\omega(N_i') + \omega(M_i') = i$. Assume, moreover, that

$$N_i' M_i' \le b_i, \quad \text{where} \quad b_i := (KB^K)^{2^i - 1}.$$

(This certainly holds when $i = 0$.) Since N and M form an amicable pair, we have

$$\frac{1}{h(N_i)h(N_i')} + \frac{1}{h(M_i)h(M_i')} = 1. \tag{2}$$

Since $\omega(N_i)+\omega(M_i)=K-i>0$, either $N_i>1$ or $M_i>1$. So either $h(\prod_{p\mid N_i}p^\infty)>h(N_i)$ or $h(\prod_{p\mid M_i}p^\infty)>h(M_i)$. Hence,

$$\begin{aligned}1 &> \frac{1}{h(\prod_{p\mid N_i}p^\infty)h(N_i')}+\frac{1}{h(\prod_{p\mid M_i}p^\infty)h(M_i')}\\ &= \frac{\prod_{p\mid N_i}(p-1)}{\prod_{p\mid N_i}p}\frac{N_i'}{\sigma(N_i')}+\frac{\prod_{p\mid M_i}(p-1)}{\prod_{p\mid M_i}p}\frac{M_i'}{\sigma(M_i')}.\end{aligned} \tag{3}$$

Letting

$$D:=\left(\prod_{p\mid N_i}p\right)\left(\prod_{p\mid M_i}p\right)\sigma(N_i')\sigma(M_i'),$$

we see that the right-hand-side of (3) is at most $1-1/D$. Comparing with (2), we deduce that either

$$\frac{1}{h(N_i)h(N_i')}-\frac{1}{h(\prod_{p\mid N_i}p^\infty)h(N_i')}\geq\frac{1}{2D}, \tag{4}$$

or that the analogous inequality holds with N_i and N_i' replaced by M_i and M_i' (respectively). The second case works out similarly to the first, so we assume that (4) holds. We find that

$$\frac{1}{2D}\leq\frac{1}{h(N_i)h(N_i')}\left(1-\frac{h(N_i)}{h(\prod_{p\mid N_i}p^\infty)}\right)\leq 1-\frac{h(N_i)}{h(\prod_{p\mid N_i}p^\infty)}=1-\prod_{p^e\| N}\left(1-\frac{1}{p^{e+1}}\right)\leq\sum_{p^e\| n}\frac{1}{p^{e+1}}.$$

In the final step, we use the inequality $\prod(1-t_j)\geq 1-\sum t_j$ valid for real numbers $0\leq t_j\leq 1$. Since the final sum on p^e has at most K terms, there must be a prime power $p^e\parallel N_i$ with

$$p^{e+1}\leq 2DK.$$

Now put $N_{i+1}:=N_i/p^e$, $N_{i+1}':=N_i\cdot p^e$, and leave M_i and M_i' unchanged. (Of course, had we been in the second case above, the roles of the Ns and Ms would be reversed.) Then $N=N_{i+1}N_{i+1}'$, $M=M_{i+1}M_{i+1}'$, $\gcd(N_{i+1},N_{i+1}')=\gcd(M_{i+1},M_{i+1}')=1$, and $\omega(N_{i+1}')+\omega(M_{i+1}')=i+1$. Moreover, the only new component introduced in N_{i+1}' or M_{i+1}' is p^e. To bound the size of p^e, notice that

$$\begin{aligned}D&\leq B^{\omega(N_i)+\omega(M_i)}\sigma(N_i')\sigma(M_i')\\ &\leq B^{\omega(N_i)+\omega(M_i)}2^{\omega(N_i')+\omega(M_i')}N_i'M_i'\leq B^K N_i'M_i',\end{aligned}$$

using in the second inequality that $\sigma(q^f)<2q^f$ for each prime power q^f. Thus,

$$p^e\leq\frac{1}{2}p^{e+1}\leq DK\leq KB^K\cdot N_i'M_i'.$$

Hence,

$$N_{i+1}'M_{i+1}'=p^eN_i'M_i'\leq(KB^K)\cdot(N_i'M_i')^2\leq KB^K\cdot b_i^2=(KB^K)^{2^{i+1}-1}=b_{i+1}.$$

This shows that if our opening assumptions hold at step i, then they continue to hold at step $i+1$. Following the induction through to the end, we obtain that

$$NM=N_K'M_K'\leq b_K=(KB^K)^{2^K-1},$$

precisely as asserted in the lemma. □

2.2. **The proof proper.** Another induction argument is needed to bound the primes dividing NM. This requires several new pieces of notation. If N_0 and M_0 are any positive integers, let

$$\mathscr{A}^{(N_0,M_0)} := \{(n,m) : N = N_0 n, M = M_0 m \text{ form a coprime amicable pair}, \\ \gcd(N_0,n) = \gcd(M_0,m) = 1\}.$$

For each integer $K \geq 0$, put

$$\mathscr{A}^{(N_0,M_0)}(K) := \{(n,m) \in \mathscr{A}^{(N_0,M_0)} : \omega(nm) = K\}.$$

For integers $0 \leq k \leq K$, set

$$\mathscr{A}^{(N_0,M_0)}(K,k) := \{(n_1,m_1) : \omega(n_1m_1) = k, \text{there exists } (n,m) \in \mathscr{A}^{(N_0,M_0)}(K) \\ \text{where } n_1 \parallel n, m_1 \parallel m, \text{ and } n_1m_1 \mid^* nm\}.$$

Put

$$\mathscr{P}^{(N_0,M_0)}(K,k) := \{p : p \mid n_1m_1 \text{ for some } (n_1,m_1) \in \mathscr{A}^{(N_0,M_0)}(K,k)\}.$$

Finally, let

$$\mathscr{R}^{(N_0,M_0)}(K,k) := \left\{\frac{1}{h(N_0)h(n_1)} + \frac{1}{h(M_0)h(m_1)} - 1 : (n_1,m_1) \in \mathscr{A}^{(N_0,M_0)}(K,k)\right\}.$$

Our plan is to obtain an explicit upper bound on $\mathscr{P}^{(N_0,M_0)}(K,k)$ valid in essentially the entire space of possible parameters N_0, M_0, K, k. Taking $N_0 = M_0 = 1$ and $k = K$ will then yield a value of B that can be inserted into Lemma 2.

In what follows, we use $R^{(N_0,M_0)}(K,k)$ for an explicit *positive* lower bound on $\mathscr{R}^{(N_0,M_0)}(K,k)$, and we write $P^{(N_0,M_0)}(K,k)$ for an explicit upper bound on $\mathscr{P}^{(N_0,M_0)}(K,k)$.

Lemma 3. *Let $K \geq 1$. Then for any N_0 and M_0, we can choose*

$$R^{(N_0,M_0)}(K,0) = \frac{1}{\sigma(N_0)\sigma(M_0)}.$$

Proof. If the set $\mathscr{A}^{(N_0,M_0)}(K,0)$ is empty, then the lemma is trivial. Otherwise, there is a coprime amicable pair of the form $N := N_0n$, $M := M_0m$ with $\gcd(N_0,n) = \gcd(M_0,m) = 1$ and $\omega(nm) = K \geq 1$. The last condition shows that either $h(N_0) < h(N)$ or $h(M_0) < h(M)$. Now $(n_1,m_1) = (1,1)$ is the only element of $\mathscr{A}^{(N_0,M_0)}(K,0)$, and

$$\begin{aligned} 0 = \frac{1}{h(N)} + \frac{1}{h(M)} - 1 &< \frac{1}{h(N_0)h(1)} + \frac{1}{h(M_0)h(1)} - 1 \\ &= \frac{N_0}{\sigma(N_0)} + \frac{M_0}{\sigma(M_0)} - 1. \end{aligned}$$

Since this final expression is a positive rational number with denominator $\sigma(N_0)\sigma(M_0)$, it is bounded below by $1/(\sigma(N_0)\sigma(M_0))$. □

Lemma 4. *Let $K \geq 1$, and suppose that $0 \leq k < K$. For any valid choice of $R^{(N_0,M_0)}(K,k)$, we can take*

$$P^{(N_0,M_0)}(K,k+1) = \frac{2K}{R^{(N_0,M_0)}(K,k)}.$$

Proof. Suppose that $(n_1,m_1) \in \mathscr{A}^{(N_0,M_0)}(K,k+1)$. Choose n_2 and m_2 coprime to N_0n_1 and M_0m_1 respectively, for which $N = N_0n_1n_2$ and $M = M_0m_1m_2$ form a coprime amicable pair, and where $n_1m_1 \mid^* n_1m_1n_2m_2$.

Let p be the largest prime dividing n_1m_1. We will assume that $p \mid n_1$; the case when $p \mid m_1$ is similar. Say $p^e \parallel n_1$. Then $(n_1/p^e, m_1) \in \mathscr{A}^{(N_0,M_0)}(K,k)$, and so

$$\frac{1}{h(N_0)h(n_1/p^e)} + \frac{1}{h(M_0)h(m_1)} - 1 \geq R^{(N_0,M_0)}(K,k).$$

On the other hand,

$$\frac{1}{h(N_0)h(n_1/p^e)h(p^e n_2)} + \frac{1}{h(M_0)h(m_1)h(m_2)} - 1 = 0.$$

Comparing the last two displays, we conclude that either

$$\frac{1}{h(N_0)h(n_1/p^e)}\left(1 - \frac{1}{h(p^e n_2)}\right) \geq \frac{1}{2}R^{(N_0,M_0)}(K,k)$$

$$\text{or}\quad \frac{1}{h(M_0)h(m_1)}\left(1 - \frac{1}{h(m_2)}\right) \geq \frac{1}{2}R^{(N_0,M_0)}(K,k).$$

In the first case, let $\ell = p^e n_2$ and in the second, let $\ell = m_2$. Since $n_1 m_1 \mid^* n_1 m_1 n_2 m_2$ and p is the largest prime factor of $n_1 m_1$, we see that every prime dividing ℓ is at least p. Thus,

$$\begin{aligned}\frac{1}{2}R^{(N_0,M_0)}(K,k) \leq 1 - \frac{1}{h(\ell)} &\leq 1 - \frac{1}{h(\prod_{q\mid\ell} q^\infty)} \\ &= 1 - \prod_{q\mid\ell}\left(1 - \frac{1}{q}\right) \leq \sum_{q\mid\ell}\frac{1}{q} \leq \frac{K}{p}.\end{aligned}$$

Hence, $p \leq 2K/R^{(N_0,M_0)}(K,k)$, as desired. □

Lemma 5. *Let $K \geq 2$. Suppose it is known that for all choices of N_0 and M_0, all integers $0 \leq k < K-1$, and all valid choices of $P^{(N_0,M_0)}(K-1,j)$ for $1 \leq j \leq k$, we may choose*

$$R^{(N_0,M_0)}(K-1,k) = \frac{1}{(2(K-1)\cdot\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K-1,j))^{2^k}}. \tag{5}$$

Then for all N_0 and M_0, all integers $0 \leq k < K$, and all valid choices of $P^{(N_0,M_0)}(K,j)$ for $1 \leq j \leq k$, we may choose

$$R^{(N_0,M_0)}(K,k) = \frac{1}{(2K\cdot\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j))^{2^k}}. \tag{6}$$

Remark. When $K = 2$, the only integer with $0 \leq k < K-1$ is $k = 0$. In this case, the product over j in (5) is empty (and to be understood as 1), and the permissibility of choosing (5) is immediate from Lemma 3.

Proof. When $k = 0$, (6) follows from Lemma 3, so we assume that $1 \leq k < K$. We let (n_1, m_1) be an arbitrary element of $\mathscr{A}^{(N_0,M_0)}(K,k)$ and we show that the right-hand-side of (6) serves as a lower bound on $\frac{1}{h(N_0)h(n_1)} + \frac{1}{h(M_0)h(m_1)} - 1$. We consider three cases based on the sign of the expresion

$$\frac{1}{h(N_0)h(\prod_{p\mid n_1} p^\infty)} + \frac{1}{h(M_0)h(\prod_{p\mid m_1} p^\infty)} - 1. \tag{7}$$

Case I. (7) is positive.

One can write (7) as a rational number with denominator $\sigma(N_0)\sigma(M_0)\prod_{p\mid n_1 m_1} p$. Stripping components off of n_1 and m_1 corresponding to the largest primes, we successively obtain elements of $\mathscr{A}^{(N_0,M_0)}(K,k-1)$, $\mathscr{A}^{(N_0,M_0)}(K,k-2)$, …, and $\mathscr{A}^{(N_0,M_0)}(K,1)$. Consequently,

$$\prod_{p\mid n_1 m_1} p \leq \prod_{j=1}^{k} P^{(N_0,M_0)}(K,j).$$

Hence,

$$\begin{aligned}\frac{1}{h(N_0)h(n_1)} + \frac{1}{h(M_0)h(m_1)} - 1 &\geq \frac{1}{h(N_0)h(\prod_{p\mid n_1} p^\infty)} + \frac{1}{h(M_0)h(\prod_{p\mid m_1} p^\infty)} - 1 \\ &\geq \frac{1}{\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j)},\end{aligned}$$

which is certainly at least as large as the right-hand side of (6).

Case II. (7) is negative.

In this case, our above estimate of the denominator in (7) yields

$$\frac{1}{h(N_0)h(\prod_{p|n_1} p^\infty)} + \frac{1}{h(M_0)h(\prod_{p|m_1} p^\infty)} - 1 \le -\frac{1}{\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j)}.$$

Since $\frac{1}{h(N_0)h(n_1)} + \frac{1}{h(N_0)h(m_1)} - 1 \ge 0$, we see that either

$$\frac{1}{h(N_0)h(n_1)}\left(1 - \frac{h(n_1)}{h(\prod_{p|n_1} p^\infty)}\right) \ge \frac{1}{2\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j)} \tag{8}$$

or that the analogous inequality holds with N_0 and n_1 replaced by M_0 and m_1. We will assume that (8) holds, since the other case is similar. Then

$$\begin{aligned}\frac{1}{2\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j)} &\le 1 - \frac{h(n_1)}{h(\prod_{p|n_1} p^\infty)}\\ &= 1 - \prod_{p^e \| n_1}\left(1 - \frac{1}{p^{e+1}}\right) \le \sum_{p^e\|n_1} \frac{1}{p^{e+1}}.\end{aligned}$$

Since there are at most K terms in the final sum, there is a prime power $p^e \parallel n_1$ with

$$p^{e+1} \le 2K\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j).$$

Now $(n_1/p^e, m_1) \in A^{(N_0p^e,M_0)}(K-1,k-1)$, and so from (5),

$$\begin{aligned}\frac{1}{h(N_0)h(n_1)} + \frac{1}{h(M_0)h(m_1)} - 1 &= \frac{1}{h(N_0p^e)h(n_1/p^e)} + \frac{1}{h(M_0)h(m_1)} - 1 \\ \ge R^{(N_0p^e,M_0)}(K-1,k-1) &\ge \frac{1}{(2(K-1)\sigma(N_0p^e)\sigma(M_0)\prod_{j=1}^{k-1} P^{(N_0p^e,M_0)}(K-1,j))^{2^{k-1}}}.\end{aligned} \tag{9}$$

It is permissible to choose $P^{(N_0p^e,M_0)}(K-1,j) = P^{(N_0,M_0)}(K,j+1)$; this comes from the existence of the well-defined map

$$\begin{aligned}\mathscr{A}^{(N_0p^e,M_0)}(K-1) &\to \mathscr{A}^{(N_0,M_0)}(K)\\ (n', m') &\mapsto (n'p^e, m'),\end{aligned}$$

and the fact that the smallest j prime factors of n' are among the smallest $j+1$ prime factors of $n'p^e$. Hence, the final denominator in (9) is at most

$$\begin{aligned}&\left(2K\cdot\sigma(N_0)\sigma(M_0)\prod_{j=2}^{k} P^{(N_0,M_0)}(K,j)\right)^{2^{k-1}} \cdot \sigma(p^e)^{2^{k-1}}\\ &\qquad\le \left(2K\cdot\sigma(N_0)\sigma(M_0)\prod_{j=2}^{k} P^{(N_0,M_0)}(K,j)\right)^{2^{k-1}}\\ &\qquad\qquad\cdot\left(2K\cdot\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j)\right)^{2^{k-1}},\end{aligned}$$

which in turn is bounded above by

$$\left(2K\cdot\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j)\right)^{2^{k}}.$$

Thus,

$$\frac{1}{h(N_0)h(n_1)} + \frac{1}{h(M_0)h(m_1)} - 1 \geq \frac{1}{(2K\cdot\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j))^{2^k}}.$$

This is again acceptable for us.

Case III. (7) vanishes.

In fact, this case never occurs. To see this, choose a coprime amicable pair $N = N_0 n$ and $M = M_0 m$ where $\gcd(N_0, n) = \gcd(M_0, m) = 1$, $n_1 \parallel n$, $m_1 \parallel m$, and $n_1 m_1 \mid^* nm$. If (7) vanishes, then multiplying through by $\sigma(N_0)\sigma(M_0)\prod_{p\mid n_1 m_1} p$ yields

$$N_0\sigma(M_0)\prod_{p\mid n_1}(p-1)\prod_{p\mid m_1} p + M_0\sigma(N_0)\prod_{p\mid m_1}(p-1)\prod_{p\mid n_1} p = \sigma(N_0)\sigma(M_0)\prod_{p\mid n_1 m_1} p.$$

Let q be the largest prime dividing $n_1 m_1$. We suppose for simplicity that $q \mid n_1$; the case when $q \mid m_1$ is exactly analogous. Looking modulo q, the previous displayed equation tells us that

$$q \mid N_0\sigma(M_0)\prod_{p\mid n_1}(p-1)\prod_{p\mid m_1} p.$$

But q is coprime to each right-hand factor except possibly $\sigma(M_0)$. Thus,

$$q \mid \sigma(M_0) \mid \sigma(M) = N + M.$$

Now $q \mid n_1 \mid n \mid N$. Since q divides $N + M$, we must have that $q \mid M$. But this contradicts that N and M are relatively prime. □

What we have shown so far has the following important consequence:

Lemma 6. *Let $K \geq 0$. If $(n,m) \in \mathscr{A}^{(N_0,M_0)}(K)$, then every prime factor of nm is bounded by*

$$(2K\cdot\sigma(N_0)\sigma(M_0))^{2^{K(K+1)/2}}.$$

Proof. This is vacuously true when $K = 0$, since then $nm = 1$. When $K = 1$, a stronger estimate is immediate from Lemmas 3 and 4 (with $k = 0$ in the latter lemma). So assume now that $K \geq 2$.

Induction on K, with $K = 2$ as the base case and with Lemma 5 providing the induction step, shows the following: *Given any $K \geq 2$, any $0 \leq k < K$, and any valid choices of $P^{(N_0,M_0)}(K,j)$ for $1 \leq j \leq k$, we may take*

$$R^{(N_0,M_0)}(K,k) = \frac{1}{(2K\cdot\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j))^{2^k}}. \tag{10}$$

From Lemmas 3 and 4, an acceptable choice for $P^{(N_0,M_0)}(K,1)$ is

$$2K\cdot\sigma(N_0)\sigma(M_0).$$

Suppose that for a given k with $1 \leq k < K$, it is acceptable to choose

$$P^{(N_0,M_0)}(K,j) = (2K\cdot\sigma(N_0)\sigma(M_0))^{2^{j(j+1)/2}} \quad \text{for all} \quad 1 \leq j \leq k.$$

(For instance, this holds when $k = 1$.) From (10) and Lemma 4, we may then choose for $P^{(N_0,M_0)}(K,k+1)$ any number of size at least

$$\begin{aligned}\frac{2K}{R^{(N_0,M_0)}(K,k)} &= (2K)\left(2K\cdot\sigma(N_0)\sigma(M_0)\prod_{j=1}^{k} P^{(N_0,M_0)}(K,j)\right)^{2^k}\\ &= (2K)^{1+2^k(1+\sum_{j=1}^{k} 2^{j(j+1)/2})}(\sigma(N_0)\sigma(M_0))^{2^k(1+\sum_{j=1}^{k} 2^{j(j+1)/2})}.\end{aligned}$$

It is straightforward to check that both exponents above are bounded by $2^{\frac{(k+1)(k+2)}{2}}$. Thus, we may take

$$P^{(N_0,M_0)}(K,k+1) = (2K\cdot\sigma(N_0)\sigma(M_0))^{2^{(k+1)(k+2)/2}}.$$

By induction on k, we may choose

$$P^{(N_0,M_0)}(K,K) = (2K \cdot \sigma(N_0)\sigma(M_0))^{2^{K(K+1)/2}}.$$

But this is exactly what is asserted by the lemma. □

We are now in a position to complete the proof our main result.

Proof of Theorem 1. In view of Hagis's result (1), we may assume that $K \geq 22$. Taking $N_0 = M_0 = 1$ in Lemma 6, we find that a coprime amicable pair N, M with $\omega(NM) = K$ has all of its prime factors bounded by $B := (2K)^{2^{K(K+1)/2}}$. By Lemma 2,

$$NM \leq K^{2^K-1}(2K)^{2^{K(K+1)/2}\cdot K\cdot(2^K-1)}.$$

This upper bound is smaller than $(2K)^{2^{K^2}}$ for all $K \geq 22$. □

Remark. While we have followed closely the argument used by Pomerance [Pom77] to bound odd perfect N with $\omega(N) = K$, our method of converting an upper bound on the prime factors to an upper bound on the numbers themselves is more efficient than the corresponding step in Pomerance's argument. (A similar improvement to Pomerance's argument was pointed out by Nielsen; see [Nie03, Proposition 1].) This explains why our upper bound is (essentially) doubly exponential in K^2, whereas Pomerance's published bound was triply exponential in K^2.

3. Concluding remarks

The theorems of Artjuhov and Borho alluded to in the introduction operate under assumptions strictly weaker than the coprimality of N and M. Here we briefly sketch how to make explicit Artjuhov's result. If N and M form an amicable pair, we define the *kernel* of N, M — denoted $\Delta(N, M)$ — to be the unitary divisor of $\mathrm{lcm}[N, M]$ supported on the primes dividing $\gcd(N, M)$. In other words,

$$\Delta(N, M) := \prod_{p \mid \gcd(N,M)} p^{\max\{v_p(N), v_p(M)\}},$$

where $v_p(\cdot)$ is the usual p-adic valuation. We prove the following:

Theorem 7. *Let N and M form an amicable pair. Suppose that $\omega(N) + \omega(M) = K$. Then*

$$NM \leq \Delta(N, M)^{2^{K+1}} K^{2^K-1}(2B)^{K\cdot(2^K-1)},$$

where

$$B := (2^{K+1}K \cdot \Delta(N, M)^2)^{2^{K(K+1)/2}}.$$

In particular, if both K and $\Delta(N, M)$ are bounded, then there are only finitely many possibilities for NM.

The main result of Artjuhov in [Art75] is precisely the final assertion of the theorem, without any bound on NM.

Let $\mathscr{S}$ be a fixed, finite set of primes. We say that two integers are *$\mathscr{S}$-coprime* if all of their common prime factors lie in $\mathscr{S}$. In place of the set $\mathscr{A}^{(N_0,M_0)}$ defined in §2.2, let

$$\tilde{\mathscr{A}}^{(N_0,M_0)} := \{(n, m) : N = N_0 n, M = M_0 m \text{ form an } \mathscr{S}\text{-coprime amicable pair},$$
$$\gcd(nm, \prod_{p\in\mathscr{S}} p) = \gcd(N_0, n) = \gcd(M_0, m) = 1\}.$$

All of the arguments of the preceding section apply essentially verbatim; we deduce in analogy with Lemma 6 that if $(n, m) \in \tilde{\mathscr{A}}^{(N_0,M_0)}$ with $\omega(n) + \omega(m) = K$, then every prime dividing nm is bounded above by

$$(2K \cdot \sigma(N_0)\sigma(M_0))^{2^{K(K+1)/2}}. \tag{11}$$

Now we note the following analogue of Lemma 2. The proof is completely analogous to our earlier argument and is omitted.

Lemma 8. *Let K and B be integers with $K \geq 0$ and $B \geq 2$. Let N_0 and M_0 be positive integers. Suppose that $N = N_0 n$ and $M = M_0 m$ form an amicable pair, where $\gcd(N_0, n) = \gcd(M_0, m) = 1$, and where $\omega(n) + \omega(m) = K$. Suppose also that every prime dividing nm is bounded above by B. Then*

$$nm \leq (KB^K \sigma(N_0)\sigma(M_0))^{2^K - 1}.$$

Combining the last two results, we obtain the following:

Corollary 9. *Let N_0 and M_0 be positive integers. Suppose that $N = N_0 n$ and $M = M_0 m$ form an $\mathscr{S}$-coprime amicable pair, where $\gcd(nm, \prod_{p \in \mathscr{S}} p) = \gcd(N_0, n) = \gcd(M_0, m) = 1$. If $\omega(n) + \omega(m) = K$, then*

$$NM \leq N_0 M_0 \cdot (KB^K \sigma(N_0)\sigma(M_0))^{2^K - 1},$$

where B is given by (11).

Proof of Theorem 7. Let $\mathscr{S}$ be the set of primes dividing both N and M, and let N_0 and M_0 be the largest divisors of N and M supported on the primes in $\mathscr{S}$. We can write $N = N_0 n$ and $M = M_0 m$, where $\gcd(N_0, n) = \gcd(M_0, m) = \gcd(nm, \prod_{p \in \mathscr{S}} p) = 1$. The stated bound follows from Corollary 9, using that $\omega(n) + \omega(m) \leq K$, that $N_0, M_0 \leq \Delta(N, M)$, and that $\sigma(N_0)\sigma(M_0) \leq 2^{\omega(N_0)+\omega(M_0)} N_0 M_0 \leq 2^K \cdot \Delta(N, M)^2$. □

We have not yet said anything about the theorem of Borho. In place of Artjuhov's strong requirement that $\Delta(N, M)$ be bounded, Borho requires only that $\Omega(\gcd(N, M))$ is bounded (in addition, of course, to the underlying assumption that $\omega(N) + \omega(M)$ is bounded). Unfortunately, it is not clear how to modify the approach taken in this paper to obtain an explicit version of Borho's theorem.

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