# Uniform and weak uniform distribution of certain arithmetic functions

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Combinatorial and Additive Number Theory

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Noah Lebowitz-Lockard

Akash Singha Roy

Definition

Let $f$ be an integer-valued arithmetic function; that is, $f$ is a function from $\mathbb{Z}_{>0}$ to $\mathbb{Z}$. Let $q$ be a positive integer. We say $f$ is **uniformly distributed modulo** $q$ (or **equidistributed mod** $q$) if, for each integer $a$,

$$\frac{1}{x}\#\{n \le x : f(n) \equiv a \pmod{q}\} \to \frac{1}{q}, \quad \text{as } x \to \infty.$$

Example (trivial): $n \mapsto n$ is equidistributed mod $q$ for every $q$.

Example (not so trivial): $n \mapsto F_n$ ($n$th Fibonacci number) is equidistributed mod $q$ if and only if $q = 5^k$ for some $k$.
(Niederreiter, Kuipers–Shiue)

Let $A(n) = \sum_{p^k \| n} kp$ be the sum of the prime factors of $n$, counted with multiplicity; e.g.,

$$A(20) = 2 + 2 + 5 = 9.$$

Theorem (Alladi–Erdős)

*$A(n)$ is equidistributed modulo 2. In fact,*

$$\sum_{n \leq x} (-1)^{A(n)} \ll x \exp(-c\sqrt{\log x \log\log x}).$$

Since primes $> 2$ are odd, $(-1)^{A(n)}$ very closely resembles the Liouville $\lambda$-function $\lambda(n) = (-1)^{\Omega(n)}$, which in turn "resembles" the classical Möbius function $\mu(n)$. This allows Alladi–Erdős to deduce their result from a known estimate for partial sums of $\mu(n)$.

**Theorem (Delange, Goldfeld)**

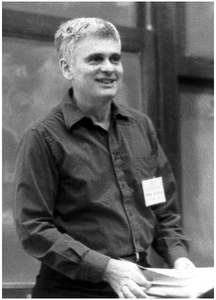
*$A(n)$ is equidistributed mod $q$ for each fixed $q$. In fact,*

$$\#\{n \le x : A(n) \equiv a \pmod{q}\} = \frac{x}{q} + O(x/\sqrt{\log x}).$$

The $O(x/\sqrt{\log x})$ error term is best possible if (e.g.) $q = 3$ but can be improved if $q$ has only large prime factors.

Approach: Work with characters of the additive group $\mathbb{Z}/q\mathbb{Z}$. This requires showing cancelation in the partial sums of the (multiplicative) function $e^{2\pi i h A(n)/q}$, for each $h$ not a multiple of $q$. Goldfeld obtains these by a version of the Landau–Selberg–Delange (LSD) method.

Hubert Delange



Dorian Goldfeld

Let $\varphi(n)$ denote Euler's totient; that is, $\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$.

It is **certainly not** the case that $\varphi(n)$ is equidistributed modulo each fixed $q$. For example,

$$\varphi(n) \quad \text{is even once } n > 2.$$

In general, $\varphi(n)$ is divisible by $q$ whenever $p \mid n$ for some prime $p \equiv 1 \pmod{q}$. For each fixed $q$, a positive proportion of primes $p$ satisfy $p \equiv 1 \bmod q$. Moreover, most numbers $n$ have many prime factors. So it should be rare for $\varphi(n)$ to **not** be 0 mod $q$.

### Proposition (Landau? Erdős?)

Fix $q$. The number of $n \leq x$ for which $\varphi(n) \not\equiv 0 \pmod{q}$ is $o(x)$, as $x \to \infty$.

## Definition (Narkiewicz)

Let $f$ be an integer-valued arithmetic function; that is, $f$ is a function from $\mathbb{Z}_{>0}$ to $\mathbb{Z}$. Let $q$ be a positive integer. We say $f$ is **weakly uniformly distributed modulo** $q$ if there are infinitely many $n$ with $\gcd(f(n), q) = 1$ and if, for each $a$ coprime to $q$,

$$\frac{\#\{n \le x : f(n) \equiv a \pmod{q}\}}{\#\{n \le x : \gcd(f(n), q) = 1\}} \to \frac{1}{\varphi(q)},$$

as $x \to \infty$.

Perhaps $\varphi(n)$ is usually weakly equidistributed mod $q$. We need $q$ odd to satisfy $\gcd(\varphi(n), q) = 1$. But this is not enough. For example,

$$\#\{n \le x : \varphi(n) \equiv 1 \pmod 3\} \sim c_1 x/\sqrt{\log x},$$

while

$$\#\{n \le x : \varphi(n) \equiv -1 \pmod 3\} \sim c_{-1} x/\sqrt{\log x},$$

whereas

$$c_1 = 0.6109\ldots, \quad c_{-1} = 0.3284\ldots$$

(see Dence and Pomerance).

Thus, we can only hope for weak equidistribution modulo $q$ when $\gcd(q, 6) = 1$.

Theorem (Narkiewicz)

*Let $q$ be any positive integer with $\gcd(q, 6) = 1$. Then $\varphi(n)$ is weakly equidistributed modulo $q$.*

What goes wrong with $q = 3$? The numbers $p - 1$, for $p$ prime and $p \neq 3$, either fail to be coprime to 3 or are "trapped" in the trivial subgroup of $(\mathbb{Z}/3\mathbb{Z})^\times$.

Approach: Work with the multiplicative (Dirichlet) characters mod $q$. One needs to show that for each $\chi$ mod $q$ that is not the trivial character $\chi_0$, one has $\sum_{n \leq x} \chi(\varphi(n)) = o(\sum_{n \leq x} \chi_0(\varphi(n)))$, as $x \to \infty$. This follows from a special case of Halász's theorem due to Wirsing (or Landau–Selberg–Delange).

For general integer-valued additive arithmetic functions, Delange has a practical necessary and sufficient condition for uniform distribution. For "polynomial-like" multiplicative functions, Narkiewicz has a practical condition for weak uniform distribution.

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For general integer-valued additive arithmetic functions, Delange has a practical necessary and sufficient condition for uniform distribution. For "polynomial-like" multiplicative functions, Narkiewicz has a practical condition for weak uniform distribution.

These generalizations can all be thought of as working in the $f$-aspect.

Question. What about the $q$-aspect? Can we prove (weak) equidistribution theorems when $q$ is allowed to vary with our stopping point $x$?

Model. The primes are weakly equidistributed mod $q$ for each fixed $q$. In fact, the primes $\leq x$ are asymptotically equidistributed in coprime residue classes mod $q$ for $q \leq (\log x)^A$ (Siegel–Walfisz).

**Theorem (Singha-Roy, P., 2022+)**

*Fix $K > 0$. As $x \to \infty$,*

$$\frac{\#\{n \le x : A(n) \equiv a \pmod q\}}{x/q} \to 1,$$

*uniformly for residue classes $a \bmod q$ with $q \le (\log x)^K$.*

**Theorem (Singha-Roy, P., 2022+)**

*Fix $K > 0$. As $x \to \infty$,*

$$\frac{\#\{n \le x : \varphi(n) \equiv a \pmod q\}}{\frac{1}{\varphi(q)}\#\{n \le x : \gcd(\varphi(n), q) = 1\}} \to 1,$$

*uniformly for coprime residue classes $a \bmod q$ with $\gcd(q, 6) = 1$ and $q \le (\log x)^K$.*

*(w/ Lebowitz-Lockard: special case $q = p$, prime)*

# A word on the proofs

What we don't do: We don't use characters! Reducing the problem to one about mean values of multiplicative functions is the right thing to do for fixed $q$. But the standard methods for estimating these sums (such as Landau–Selberg–Delange, or quantitative versions of Halász's theorem) seem to yield the desired asymptotics only in much more limited ranges of $q$.

Instead, we develop a quasi-elementary method suggested by work of Banks–Harman–Shparlinski, who proved a theorem of the same kind on the weak-equidistribution of $P^{+}(n)$, the largest prime factor of $n$.

Here is an outline, suppressing unpleasant details. The case of $A(n)$ is the simpler one. We make things even simpler by restricting to $q$ odd.

Initial reduction step. Erdős-ian trickery allow us to discard inconvenient $n$. Let $J = J(x)$ be an integer that tends to infinity but very slowly, say

$$J = \lfloor \log\log\log x \rfloor.$$

Call $n$ **convenient** if the $J$ largest prime factors of $n$ (with multiplicity), say

$$P_J \le P_{J-1} \le \cdots \le P_1,$$

are all at least $y = \exp(\sqrt{\log x})$.

This choice of threshold $y$ is chosen so that past $y$, primes are very regularly distributed in coprime residue classes mod $q$, when $q \le (\log x)^K$.

We can reduce to proving that for each residue class $a$ mod $q$,

$$\frac{\#\{n \le x : n \text{ convenient}, A(n) \equiv a \pmod q\}}{\#\{n \le x : n \text{ convenient}\}} \sim \frac{1}{q}.$$

The denominator here is $\sim x$, but we leave it as is for a reason: Rather than estimate the numerator directly, we will make a direct comparison with number of convenient $n \le x$.

Write

$$n = mP_1 \dots P_J,$$

so that $A(n) = A(m) + P_1 + P_2 + \cdots + P_J$.

Let $N$ be the count of convenient $n \le x$. Then

$$N := \sum_{\substack{n \le x \\ n \text{ convenient}}} 1 = \sum_{m \le x} {\sum}'_{P_1,\dots,P_J} 1$$

where the $'$ conditions on the primes $P_1, \dots, P_J$ are that

$$\max\{P^+(m), y\} \le P_J \le P_{J-1} \le \cdots \le P_2 \le P_1 \le x,$$

$$P_1 \cdots P_J \le x/m.$$

For each $m \le x$, let $V(m)$ be the set of all $J$-tuples $(v_1, \dots, v_J)$ of coprime residue classes mod $q$ for which

$$v_1 + \cdots + v_J \equiv a - A(m) \pmod{q}.$$

Then

$$\sum_{\substack{n \le x \\ n \text{ convenient} \\ A(n) \equiv a \pmod{q}}} 1 = \sum_{m \le x} {\sum}''_{P_1,\dots,P_J} 1$$

where the $''$ conditions on $P_1, \dots, P_J$ are that

$$\max\{P^+(m), y\} \le P_J \le P_{J-1} \le \cdots \le P_2 \le P_1 \le x,$$

$$P_1 \cdots P_J \le x/m,$$

$$(P_1, \dots, P_J) \pmod{q} \in V(m).$$

Continuing...

$$\sum_{\substack{n\le x\\ n \text{ convenient}\\ A(n)\equiv a \pmod q}} 1 = \sum_{m\le x} {\sum}''_{P_1,\dots,P_J} 1$$

Carefully applying Siegel–Walfisz, one shows that

$$\sum_{m\le x} {\sum}''_{P_1,\dots,P_J} 1 \approx \sum_{m\le x} \frac{\#V(m)}{(\varphi(q))^J} {\sum}'_{P_1,\dots,P_J} 1.$$

But $\#V(m)$ can be estimated by some fairly straightforward combinatorial and additive number theory; one finds that $\#V(m)/\varphi(q)^J \approx 1/q$, uniformly in $m$ and $a$. (This uses that $q$ is odd!) Thus, the RHS in the last display is

$$\approx \frac{1}{q} \sum_{m\le x} {\sum}'_{P_1,\dots,P_J} 1 = \frac{1}{q} N.$$

The proof for $\varphi$ follows the same general plan but the estimates require more fancy footwork, mostly from the of the *anatomy of integers* and *sieve methods* repertoires.

It is helpful for us that Scourfield, in the mid-80s, already estimated $\#\{n \leq x : \gcd(\varphi(n), q) = 1\}$ fairly precisely, in a wide range of $q$. Our "comparison method" does not need too sharp an estimate, but the handling of error terms (suppressed in the sketch above) needs *something*, and Scourfield's results are more than enough.

Thank you for your attention!