CLUSTERING OF LINEAR COMBINATIONS OF MULTIPLICATIVE FUNCTIONS

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Abstract. A real-valued arithmetic function F is said to *cluster* about the point $u \in \mathbb{R}$ if the upper density of n with $u - \delta < F(n) < u + \delta$ is bounded away from 0, uniformly for all $\delta > 0$. We establish a simple-to-check sufficient condition for a linear combination of multiplicative functions to be *nonclustering*, meaning not clustering anywhere. This provides a means of generating new families of arithmetic functions possessing continuous distribution functions. As a specific application, we resolve a problem posed recently by Luca and Pomerance.

1. Introduction

Let F be a real-valued arithmetic function. We say that F *clusters around the real number* u if there is some $\epsilon > 0$ such that, for every $\delta > 0$, the solutions n to

$$u - \delta < F(n) < u + \delta$$

form a set of upper density at least ϵ. If F does not cluster around any u, we say that F is *nonclustering*. The main result of this note is the following criterion for a linear combination of multiplicative functions to be nonclustering.

Theorem 1. *Let $f_1, \dots, f_k$ be multiplicative arithmetic functions taking values in the nonzero real numbers and satisfying the following conditions:*

(i) *f_1 is nonclustering,*
(ii) *none of $f_1, \dots, f_k$ cluster around 0,*
(iii) *for all $i < j$ with $i, j \in \{1, 2, \dots, k\}$, the function f_i/f_j is nonclustering.*

Then for all nonzero $c_1, \dots, c_k \in \mathbb{R}$, the arithmetic function $F := c_1 f_1 + \cdots + c_k f_k$ is nonclustering.

Theorem 1 has consequences for the study of limit laws of arithmetic functions (for background, see, e.g., [14, Chapters III.2 and III.4] and [12, Chapter 4]). It is easy to see that for an arithmetic function F possessing a limit law (i.e., possessing a distribution function), the distribution function is continuous precisely when F is nonclustering. Now it is often the case that one can prove a distribution function exists by some general principle, but that the proof does not offer any insight into whether that function is continuous. Theorem 1 sometimes provides a convenient way of establishing continuity.

We illustrate by proving a recent conjecture of Luca and Pomerance. Let $s(n)$ be the sum-of-proper-divisors function, so that $s(n) = \sigma(n) - n$. Let $s_\phi(n) = n - \phi(n)$ denote the

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cototient function. In [10], Luca and Pomerance noted that $s(n)/s_\phi(n) \geq 1$ for all $n \geq 2$ and showed that the sequence $\{s(n)/s_\phi(n)\}_{n=2}^\infty$ is dense in $[1,\infty)$. We prove:

Theorem 2. *The arithmetic function $s(n)/s_\phi(n)$ possesses a continuous distribution function D_{s/s_ϕ}. Moreover, $D_{s/s_\phi}(u)$ is strictly increasing for $u \geq 1$.*

Theorem 2 was conjectured at the end of [10, §1].

2. Nonclustering of $c_1 f_1 + \cdots + c_k f_k$: Proof of Theorem 1

Our argument is modeled on work of Galambos and Kátai [6] concerning pairs of additive functions (generalizing an earlier result of Fein and Shapiro [5]).

2.1. **Setup.** Since f_1 is nonclustering and c_1 is nonzero, the theorem is obvious when $k = 1$. Proceeding inductively, we may assume that $k \geq 2$ and that the theorem is already known to hold for all smaller values of k.

Let $u \in \mathbb{R}$. We will show that by making a judicious choice of δ, the upper density of the set of n satisfying

$$u - \delta < F(n) < u + \delta \tag{1}$$

can be made arbitrarily small.

Let $\epsilon > 0$. We let Y and Z be large, fixed real numbers (independent of n); their values will be specified more precisely in the course of the proof. To begin with, we assume that $Y, Z \geq 2$.

For each solution n to (1), we split off the Y-smooth part of n, writing

$$n = st, \qquad \text{where} \quad p \mid s \implies p \leq Y, \quad \text{and} \quad p \mid t \implies p > Y.$$

(Here and below, p always refers to a prime.) We refer to this way of writing n as the 'basic decomposition', and we reserve the letters s and t for this purpose. We sometimes make use of obvious modifications of this notation, e.g., using s' and t' for the components in the decomposition of n'.

For a set $\mathcal{S}$ of positive integers, we write $\bar{\mathbf{d}}\mathcal{S}$ for its upper density.

2.2. **Those n with large smooth part.** It is known that the upper density of n with Y-smooth part larger than Y^Z is

$$\ll \exp(-cZ),$$

where $c > 0$ is an absolute constant, and the implied constant is also absolute (see [8, Theorem 07, p. 4]). Hence, this same expression bounds the upper density of solutions n to (1) with $s > Y^Z$.

2.3. **Splitting the set of remaining n.** Let $\mathcal{S}$ be the set of n satisfying (1) with $s \leq Y^Z$. We split $\mathcal{S}$ into two pieces, $\mathcal{S}_1$ and $\mathcal{S}_2$, where

$$\mathcal{S}_1 = \{n \in \mathcal{S} : \text{there is an } n' \in \mathcal{S} \text{ with } t = t' \text{ and with } f_i(s) \neq f_i(s') \text{ for some } i\},$$
$$\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1.$$

We proceed to bound the upper densities of $\mathcal{S}_1$ and $\mathcal{S}_2$.

2.4. **Bounding $\bar{\mathbf{d}}\mathcal{S}_1$.** Let $n \in \mathcal{S}_1$, and choose n' as in the definition of $\mathcal{S}_1$. Since n and n' both satisfy (1),

$$|F(n) - F(n')| = \Big|\sum_{i=1}^{k} c_i f_i(n) - \sum_{i=1}^{k} c_i f_i(n')\Big| < 2\delta.$$

Writing $f_i(n) = f_i(s)f_i(t)$, $f_i(n') = f_i(s')f_i(t')$ and keeping in mind that $t = t'$, the preceding inequality becomes

$$\Big|\sum_{i=1}^{k} c_i(f_i(s) - f_i(s'))f_i(t)\Big| < 2\delta.$$

Let $r = r(n)$ be the largest index in $\{1, 2, \ldots, k\}$ with $f_r(s) \neq f_r(s')$. Then

$$\Big|\sum_{i=1}^{r-1} c_i(f_i(s) - f_i(s'))\frac{f_i}{f_r}(t) + c_r(f_r(s) - f_r(s'))\Big| < \frac{2}{|f_r(t)|}\delta.$$

Since none of $f_1, \ldots, f_k$ cluster around 0, we may select $\rho > 0$ (depending on the f_i, ϵ, Y, and Z) in such a way that the set $\mathcal{T}$ of positive integers m satisfying $|f_i(m)| < \rho$ for some i has upper density less than ϵY^{-Z}. If $|f_r(t)| < \rho$, then $t = n/s \in \mathcal{T}$, and so $n \in s\mathcal{T}$. For each s,

$$\bar{\mathbf{d}}(s\mathcal{T}) = \frac{1}{s}\bar{\mathbf{d}}(\mathcal{T}) \leq \bar{\mathbf{d}}(\mathcal{T}) < \epsilon Y^{-Z}.$$

But the number of possibilities for s is at most Y^Z. Thus, the set of $n \in \mathcal{S}_1$ with $|f_r(t)| < \rho$ has upper density at most ϵ.

Suppose now that $n \in \mathcal{S}_1$ and that $|f_r(t)| \geq \rho$. Then continuing the above calculation,

$$\Big|\sum_{i=1}^{r-1} c_i(f_i(s) - f_i(s'))\frac{f_i}{f_r}(t) + c_r(f_r(s) - f_r(s'))\Big| < \frac{2}{\rho}\delta. \tag{2}$$

We enforce the condition that $\delta > 0$ is small enough that

$$\frac{2}{\rho}\delta < \min_{1 \leq i \leq k} \min_{\substack{S, S' \leq Y^Z \\ f_i(S) \neq f_i(S')}} |c_i(f_i(S) - f_i(S'))|.$$

Then (2) implies that there is at least one value of $i \in \{1, 2, \ldots, r-1\}$ with $f_i(s) \neq f_i(s')$. We now apply the induction hypothesis to the list of functions f_i/f_r, where i runs over those indices not exceeding $r - 1$ for which $f_i(s) \neq f_i(s')$. (It is easy to see that condition (iii) for the original list $f_1, \ldots, f_k$ implies all of conditions (i)–(iii) for the new list of functions f_i/f_r.) This induction hypothesis implies that

$$\sum_{i=1}^{r-1} c_i(f_i(s) - f_i(s'))\frac{f_i}{f_r}$$

does not cluster around $-c_r(f_r(s) - f_r(s'))$. We may thus fix $\delta_{r,s,s'} > 0$ small enough to guarantee that the set $\mathcal{U}_{r,s,s'}$ of positive integers m satisfying

$$\Big|\sum_{i=1}^{r-1} c_i(f_i(s) - f_i(s'))\frac{f_i}{f_r}(m) + c_r(f_r(s) - f_r(s'))\Big| < \frac{2}{\rho}\delta_{r,s,s'}$$

has upper density smaller than $\epsilon Y^{-2Z} k^{-1}$. We make the further stipulation that our choice of $\delta > 0$ satisfies

$$\delta < \min \delta_{r,s,s'}$$

where the minimum runs over all of the (finitely many!) possible triples r, s, s' that arise in this way.

With δ so restricted, whenever (2) holds, $n \in s\mathcal{U}_{r,s,s'}$. Each set $s\mathcal{U}_{r,s,s'}$ has upper density smaller than $\epsilon Y^{-2Z} k^{-1}$, while the number of possibilities for the triple r, s, s' is at most kY^{2Z}. Hence, the set of $n \in \mathcal{S}_1$ with $|f_r(t)| \geq \rho$ has upper density smaller than ϵ.

We conclude that $\mathcal{S}_1$ has upper density smaller than 2ϵ.

2.5. **Bounding $\bar{\mathbf{d}}\mathcal{S}_2$.** For each large real number x, we partition $\mathcal{S}_2 \cap [1, x]$ as follows. Given a pair of nonnegative integers U, V, we let $\mathcal{S}_2(U, V)$ be the subset of $\mathcal{S}_2 \cap [1, x]$ consisting of those n with

$$x/2^{U+1} < n \leq x/2^U \quad \text{and} \quad x/2^{(U+1)+V} < t \leq x/2^{U+V}.$$

Thus,

$$\mathcal{S}_2 \cap [1, x] = \bigcup_{U,V \geq 0} \mathcal{S}_2(U, V).$$

If $n \in \mathcal{S}_2(U, V)$, then

$$2^{V-1} < s = n/t < 2^{V+1}.$$

Since each $n \in \mathcal{S}_2$ has $s \leq Y^Z$, the set $\mathcal{S}_2(U, V)$ is empty unless $2^{V-1} < Y^Z$, and so we will assume this condition on V. To bound $\#\mathcal{S}_2(U, V)$, we first fix the large-primes component t and count the number of corresponding n. List these as

$$n_1 = s_1 t, \quad n_2 = s_2 t, \quad \ldots, \quad n_J = s_J t.$$

Then for each $1 \leq i \leq k$,

$$f_i(s_1) = f_i(s_2) = \cdots = f_i(s_J);$$

otherwise, some of $n_1, \ldots, n_J$ would belong to $\mathcal{S}_1$. In particular, every $n \in \mathcal{S}_2(U, V)$ corresponding to this particular t has

$$f_1(s) = d$$

for a fixed d. By a theorem of Halász, the number of positive integers $S < 2^{V+1}$ with $f_1(S) = d$ is

$$\ll 2^{V+1}/\sqrt{E(2^{V+1})} \tag{3}$$

with an absolute implied constant, where $E(T)$ is defined for real values of T by

$$E(T) = \sum_{\substack{p \leq T \\ f_1(p) \neq \pm 1}} \frac{1}{p}.$$

(To deduce this from the main theorem of [7], apply that result to the additive function $\log |f_1(n)|$.) Our hypothesis that f_1 is nonclustering implies that the unrestricted sum

$\sum_{p\colon f_1(p)\neq\pm1}\frac{1}{p}$ diverges: Otherwise, the set of squarefree n divisible only by primes p with $f_1(p)=\pm1$ has density

$$\prod_{p\colon f_1(p)\neq\pm1}\left(1-\frac{1}{p}\right)\prod_{p\colon f_1(p)=\pm1}\left(1-\frac{1}{p^2}\right)>0,$$

which forces f_1 to cluster around one of ±1. Hence, the denominator in (3) tends to infinity with V. Thus, there is a positive integer $V_0=V_0(\epsilon)$ such that whenever $V\geq V_0$, the number of $S<2^{V+1}$ satisfying $f_1(S)=d$ is at most $\epsilon\cdot 2^{V+1}$. (We could also have reached this conclusion by applying [3, Theorem IV] instead of [7].) We conclude that, for each fixed t, the number of corresponding $n=st\in\mathcal{S}_2(U,V)$ is

$$\leq\begin{cases}2^{V+1} & \text{always},\\ \epsilon\cdot 2^{V+1} & \text{when } V\geq V_0.\end{cases}$$

On the other hand, since $t\leq x/2^{U+V}$ and has no prime factors in $[2,Y]$, inclusion-exclusion shows that the number of possibilities for t is

$$\leq\frac{x}{2^{U+V}}\prod_{p\leq Y}\left(1-\frac{1}{p}\right)+O(2^Y)\leq\frac{x}{2^{U+V}\log Y}+O(2^Y).$$

Combining these upper bounds, we deduce that

$$\#\mathcal{S}_2(U,V)\leq\begin{cases}\frac{2x}{2^U\log Y}+O(2^{V+Y}) & \text{always},\\ \frac{2\epsilon x}{2^U\log Y}+O(2^{V+Y}) & \text{for } V\geq V_0.\end{cases}$$

Finally we sum over U and V. Let $\mathcal{S}_2(U)=\bigcup_V\mathcal{S}_2(U,V)$. Since we need only consider V with $2^{V-1}<Y^Z$, we have

$$\begin{aligned}\#\mathcal{S}_2(U)\leq&\sum_{\substack{0\leq V<V_0\\ V<\frac{\log(Y^Z)}{\log 2}+1}}\left(\frac{2x}{2^U\log Y}+O(2^{V+Y})\right)+\sum_{\substack{V\geq V_0\\ V<\frac{\log(Y^Z)}{\log 2}+1}}\left(\frac{2\epsilon x}{2^U\log Y}+O(2^{V+Y})\right)\\ \leq&\frac{2V_0}{\log Y}\frac{x}{2^U}+4\epsilon Z\frac{x}{2^U}+O(2^Y\cdot Y^Z).\end{aligned}$$

Now we sum on all nonnegative U with $2^U\leq x$ to find that

$$\#\mathcal{S}_2\cap[1,x]\leq\frac{4V_0}{\log Y}\cdot x+8\epsilon Z\cdot x+O(2^Y\cdot Y^Z\cdot\log x).$$

It follows that $\mathcal{S}_2$ has upper density at most

$$\frac{4V_0}{\log Y}+8\epsilon Z.$$

2.6. **Denouement.** Putting everything together, we see that the upper density of solutions to (1) is at most

$$C\exp(-cZ)+2\epsilon+\frac{4V_0}{\log Y}+8\epsilon Z,$$

where C and c are absolute positive constants. We now fix our choices of parameters ϵ, Y, Z. Given any $\eta>0$, we first fix Z large enough to make $C\exp(-cZ)<\eta/3$, then fix

$\epsilon > 0$ small enough to make $2\epsilon + 8\epsilon Z < \eta/3$, and then finally fix Y large enough to make $4V_0/\log Y < \eta/3$. Our arguments then show that for a suitable of choice of $\delta > 0$, the set of n satisfying (1) has upper density $< \eta$.

3. s vs s_ϕ: Proof of Theorem 2

We begin with a result of independent interest.

Proposition 3. *Fix a nonzero real number R. Then $F(n) = \frac{\sigma(n)}{n} + R\frac{\phi(n)}{n}$ possesses a continuous distribution function.*

Proof that a (possibly discontinuous) distribution function exists. We argue via the method of moments. The argument is very similar to one described in detail in [11, §4], and so we only sketch the proof. For each positive integer k, define

$$\mu_k = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \left(\frac{\sigma(n)}{n} + R\frac{\phi(n)}{n}\right)^k.$$

To see that μ_k exists, it suffices to note that

$$(\sigma(n)/n + R\phi(n)/n)^k = \sum_{j=0}^{k} \binom{k}{j} R^{k-j} \sigma(n)^j \phi(n)^{k-j}/n^k$$

and that each of the functions $\sigma(n)^j\phi(n)^{k-j}/n^k$ possesses a finite mean value, by a straightforward application of Wintner's mean value theorem [12, Theorem 1, p. 138]. Since

$$\binom{k}{j} \le 2^k \qquad \text{and} \qquad \sigma(n)^j\phi(n)^{k-j}/n^k \le (\sigma(n)/n)^k \le (n/\phi(n))^k,$$

we can use the estimation of the moments of $n/\phi(n)$ appearing in the proof of [11, Proposition 4.3] to deduce that

$$\mu_k \ll \exp(O(k \log\log(3k))).$$

(Here we allow implied constants to depend on R.) In particular, the condition

$$\limsup_{k\to\infty} \mu_{2k}^{1/2k}/k < \infty$$

that is required for application of [2, Theorem 3.3.12, p. 123] is satisfied, and so $F(n)$ possesses a distribution function. □

Proof of continuity. We apply Theorem 1 with $f_1(n) = \sigma(n)/n$ and $f_2(n) = \phi(n)/n$. The Erdős–Wintner theorem [4] (see also [12, §4.7]), applied to $\log f_1$, $\log f_2$, and $\log(f_1/f_2)$ shows that all of $f_1, f_2, f_1/f_2$ have continuous distribution functions, which immediately implies conditions (i)–(iii). □

Remark 4. Results closely related to Proposition 3 can already be found in the literature. For example, [9] contains a proof of the continuity of the distribution function of $\frac{\sigma(n)}{n} + \frac{\phi(n)}{n}$ in a strong form (a sharp estimate for the modulus of continuity). The strength of Theorem

1 is its ease of applicability and wide generality. To illustrate with a random example, an argument analogous to the above will prove that

$$c_1\frac{\phi(n)}{\sigma(n)} + c_2 \exp\left(\sum_{p|n}\frac{1}{\log p}\right) + c_3\frac{\sigma(n)\lambda(n)}{n}$$

has a continuous distribution function for any nonzero c_1, c_2, c_3. Here σ, ϕ are as usual, and λ is the Liouville function, the completely multiplicative function with $\lambda(p) = -1$ for every prime p. (To estimate the moments in this case one should appeal to [15, Sätze I, II] in place of Wintner's theorem.)

Proof of Theorem 2. Let $u > 0$. Writing $s(n) = \sigma(n) - n$ and $s_\phi(n) = n - \phi(n)$, the inequality $s(n)/s_\phi(n) \leq u$ can be put in the form

$$\frac{\sigma(n)}{n} + u\frac{\phi(n)}{n} \leq 1 + u.$$

By Proposition 3, $\frac{\sigma(n)}{n} + u\frac{\phi(n)}{n}$ possesses a continuous distribution function, say $D_{1,u}$. It follows that, for each $u > 0$,

$$\lim_{x\to\infty}\frac{1}{x}\#\{2 \leq n \leq x : s(n)/s_\phi(n) \leq u\}$$

exists and equals $D_{1,u}(1 + u)$. Since $s(n)/s_\phi(n) \geq 1$, the same limit also exists for $u \leq 0$, where it vanishes. We denote the value of this limit by $D_{s/s_\phi}(u)$.

We now check the boundary conditions necessary for D_{s/s_ϕ} to qualify as a distribution function. It is trivial that $\lim_{u\to-\infty} D_{s/s_\phi}(u) = 0$. To see that $\lim_{u\to\infty} D_{s/s_\phi}(u) = 1$, suppose that $s(n)/s_\phi(n) > u$, where u is large and positive. We can write this inequality in the form

$$\frac{\frac{\sigma(n)}{n} - 1}{1 - \frac{\phi(n)}{n}} > u.$$

So either $\frac{\sigma(n)}{n} > 1 + \sqrt{u}$ or $\frac{\phi(n)}{n} > 1 - \frac{1}{\sqrt{u}}$. Each of these inequalities holds on a set of density tending to 0 as $u \to \infty$, since $\frac{\sigma(n)}{n}$ and $\frac{\phi(n)}{n}$ each have continuous distribution functions (e.g., by the Erdős–Wintner theorem again). It follows that $1 - D_{s/s_\phi}(u) \to 0$ as $u \to \infty$, and hence $D_{s/s_\phi}(u) \to 1$ as $u \to \infty$, as desired.

Now we show continuity of $D_{s/s_\phi}(u)$. It is certainly sufficient to consider values of $u \geq 1$. Given such a u, we will prove that the set of solutions n to

$$u - \delta < \frac{s(n)}{s_\phi(n)} < u + \delta$$

comprise a set of upper density tending to 0 as $\delta \downarrow 0$. Therefore s/s_ϕ is nonclustering (provided one extends this quotient to be defined at $n = 1$). Rearranging these inequalities for $s(n)/s_\phi(n)$ yields

$$\frac{\sigma(n)}{n} + u\frac{\phi(n)}{n} \leq 1 + u + \delta\left(1 - \frac{\phi(n)}{n}\right) \leq 1 + u + \delta$$

as well as

$$\frac{\sigma(n)}{n} + u\frac{\phi(n)}{n} \geq 1 + u - \delta\left(1 - \frac{\phi(n)}{n}\right) \geq 1 + u - \delta.$$

Now the desired result follows from the continuity of the distribution function $D_{1,u}$.

So far we have shown that s/s_ϕ has a continuous distribution function D_{s/s_ϕ}. It remains (only) to prove that $D_{s/s_\phi}(u)$ is strictly increasing for $u \ge 1$.

We let $a, b \ge 1$ with $a < b$ and aim to show that $D_{s/s_\phi}(a) < D_{s/s_\phi}(b)$. By [10], the image of s/s_ϕ is dense in $[1, \infty)$, and so we may fix an n_0 such that

$$c := s(n_0)/s_\phi(n_0) \in (a, b).$$

We now argue that a positive proportion of the multiples n of n_0 also satisfy $s(n)/s_\phi(n) \in (a, b)$. It is easy to prove (see the start of [10, §3]) that

$$s(n_0m)/s_\phi(n_0m) \ge s(n_0)/s_\phi(n_0) > a$$

for all m, and so it suffices to show that $s(n_0m)/s_\phi(n_0m) < b$ holds a positive proportion of the time.

Let y be a large, fixed real parameter be specified more precisely below. To begin with, we assume y is so large that $\prod_{p\le y}(1 - 1/p) > 1/(2 \log y)$. (This is true for all large y by Mertens' theorem, since $e^\gamma < 2$.) Let P_y be the product of the primes not exceeding y. Then for all sufficiently large x (depending on y),

$$\#\{m \le x : \gcd(m, P_y) = 1\} > \frac{1}{2}x \prod_{p\le y}(1 - 1/p) > \frac{1}{4 \log y}x. \tag{4}$$

Moreover, recalling that $\frac{\sigma(m)}{m} = \sum_{d|m} \frac{1}{d}$, we have that

$$\begin{aligned}\sum_{\substack{m\le x\\ \gcd(m,P_y)=1}} \left(\frac{\sigma(m)}{m} - 1\right) &= \sum_{\substack{m\le x\\ \gcd(m,P_y)=1}} \sum_{\substack{d|m\\ d>1}} \frac{1}{d} \le \sum_{\substack{d:\ p|d \implies p>y\\ d>1}} \frac{1}{d} \sum_{\substack{m\le x\\ d|m}} 1\\ &\le x \sum_{\substack{d:\ p|d \implies p>y\\ d>1}} \frac{1}{d^2} = x\left(\prod_{p>y}\left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \dots\right) - 1\right).\end{aligned}$$

The prime number theorem together with partial summation implies that

$$\begin{aligned}\prod_{p>y}\left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \dots\right) &< \exp\left(\sum_{p>y} \frac{2}{p^2}\right)\\ &\le \exp\left(O\left(\frac{1}{y \log y}\right)\right) = 1 + O\left(\frac{1}{y \log y}\right).\end{aligned}$$

Hence,

$$\sum_{\substack{m\le x\\ \gcd(m,P_y)=1}} \left(\frac{\sigma(m)}{m} - 1\right) \ll \frac{1}{y \log y}x,$$

so that the number of $m \le x$ with $\gcd(m, P_y) = 1$ and $\frac{\sigma(m)}{m} - 1 \ge \frac{1}{\log y}$ is $O(x/y)$. Comparing with (4), we see that if y is fixed sufficiently large, then for all large x,

$$\#\Big\{m \le x : \gcd(m, P_y) = 1, \frac{\sigma(m)}{m} - 1 < \frac{1}{\log y}\Big\} > \frac{1}{8 \log y}x. \tag{5}$$

Increasing y if necessary, we may assume that y exceeds the largest prime factor of n_0. Then for any m counted on the left-hand side of (5),

$$\frac{\sigma(n_0m)}{n_0m}-1=\frac{\sigma(n_0)}{n_0}\frac{\sigma(m)}{m}-1\le\frac{\sigma(n_0)}{n_0}\left(1+\frac{1}{\log y}\right)-1=\frac{\sigma(n_0)}{n_0}-1+\frac{\sigma(n_0)/n_0}{\log y}.$$

Since also

$$1-\frac{\phi(n_0m)}{n_0m}\ge 1-\frac{\phi(n_0)}{n_0},$$

we find that

$$\begin{aligned}\frac{s(n_0m)}{s_\phi(n_0m)}=\frac{\frac{\sigma(n_0m)}{n_0m}-1}{1-\frac{\phi(n_0m)}{n_0m}}&\le\frac{\frac{\sigma(n_0)}{n_0}-1}{1-\frac{\phi(n_0)}{n_0}}+\frac{\sigma(n_0)/n_0}{(1-\frac{\phi(n_0)}{n_0})}\frac{1}{\log y}\\&=c+\frac{\sigma(n_0)/n_0}{(1-\frac{\phi(n_0)}{n_0})}\frac{1}{\log y}.\end{aligned}$$

Increasing y if necessary, we can ensure that this last expression is smaller than b.

With y fixed as above, (5) implies that the set of m with $s(n_0m)/s_\phi(n_0m)<b$ has positive lower density. It follows that the corresponding values $n=n_0m$ also comprise a set of positive lower density. Together with our earlier remarks, we conclude that $D_{s/s_\phi}(a)<D_{s/s_\phi}(b)$, as desired. This completes the proof that D_{s/s_ϕ} is increasing as well as the proof of Theorem 2. □

4. Concluding remarks on positive-valued multiplicative functions

Theorem 1 is well-suited to proving the continuity of a distribution function when it exists. It is therefore natural to ask for a general condition guaranteeing that $F=c_1f_1+\cdots+c_kf_k$ possesses a distribution function. We conclude by sketching a proof of the following partial result in this direction. The argument is due essentially to Shapiro [13] (see especially p. 63), but as the case we work in is much simpler than his general set-up, it seems a relatively self-contained discussion is warranted.

Proposition 5. *Let $f_1,\dots,f_k$ be positive-valued multiplicative functions each possessing a distribution function. Then for any $c_1,\dots,c_k\in\mathbb{R}$, the function $c_1f_1+\cdots+c_kf_k$ also has a distribution function.*

Note that this result applies, for instance, to the example considered in Proposition 3, but not immediately to the one considered in Remark 4.

Let $Y>0$. We keep the notation of §2, where n denotes a positive integer and s denotes the Y-smooth part of n. (There will be no confusion with the sum-of-proper-divisors function.) We say that an arithmetic function F is *essentially determined by small primes* if for all $\epsilon>0$,

$$\lim_{Y\to\infty}\bar{\mathbf{d}}\{n:|F(n)-F(s)|>\epsilon\}=0.$$

If F is an arithmetic function essentially determined by small primes, then F has a distribution function; this is contained in [13, Theorem 2.1], and also follows from [14, Theorem 2.3, p. 427]. Moreover, the converse holds for all additive functions F (see the theorem stretching from pp. 719–720 in [4]).

To relate this back to Proposition 5, we recall that when a positive-valued multiplicative function possesses a limit law, either its distribution function is that of the degenerate distribution at 0, or the additive function $\log f$ has a distribution function. (See [1, Theorem 4], and note that the convergence of the three series in eq. (3) there is exactly the Erdős–Wintner condition for $\log f$ to have a distribution function.) Now given $f_1, \dots, f_k$ as in Proposition 5, we may reorder the list so that $f_1, \dots, f_\ell$ have distributions degenerate at 0, and $f_{\ell+1}, \dots, f_k$ do not. It is then easy to see that if $c_{\ell+1}f_{\ell+1} + \dots + c_k f_k$ has a distribution function, then $c_1 f_1 + \dots + c_k f_k$ has the same distribution function. Thus, we can (and do) assume that each of the $\log f_i$ has a distribution function. As discussed in the previous paragraph, this means that each $\log f_i$ is esssentially determined by small primes. We claim that each f_i is also so-determined. Indeed, suppose that

$$|f_i(n) - f_i(s)| > \epsilon.$$

Then, with $\eta > 0$ a parameter at our disposal, either $f_i(n) > \eta$, or

$$|f_i(s)/f_i(n) - 1| > \epsilon/\eta.$$

This last inequality implies that

$$|\log f_i(n) - \log f_i(s)| \gg_{\epsilon,\eta} 1;$$

since $\log f_i$ is essentially determined by small primes, this estimate holds on a set of upper density tending to 0 as $Y \to \infty$. On the other hand, if $f_i(n) > \eta$, then $\log f_i(n) > \log \eta$. That occurs on a set of upper density tending to 0 as η tends to infinity, since $\log f_i$ has a (proper) distribution function. Letting $Y \to \infty$ and then letting $\eta \to \infty$ proves our claim.

Since the f_i are essentially by determined by small primes, so is any $\mathbb{R}$-linear combination of the f_i; thus, all such combinations possess distribution functions.

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