Contemporary Mathematics

Eigenvalues of the Laplacian on domains with fractal boundary

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For Michel Lapidus on his 60th birthday

Abstract. Consider the Laplacian operator on a bounded open domain in Euclidean space with Dirichlet boundary conditions. We show that for each number D with $1 < D < 2$, there are two bounded open domains in $\mathbf{R}^2$ of the same area, with their boundaries having Minkowski dimension D, and having the same content, yet the secondary terms for the eigenvalue counts are not the same. This was shown earlier by Lapidus and the second author, but a possible countable set of exceptional dimensions D were excluded. Here we show that the earlier construction has no exceptions.

1. Introduction

Let Ω be a nonempty, bounded open set in $\mathbf{R}^2$. We consider eigenvalues for the Laplacian operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ for the closure in the Sobolev space of smooth functions with compact support, which are 0 on $\partial\Omega$. It is well-known that the nonzero eigenvalues are negative, forming a discrete multiset, with each multiplicity bounded. By convention we consider the absolute value of these eigenvalues and label them $0 < \lambda_1 \le \lambda_2 \le \dots$. Let

$$N(\lambda; \Omega) = \sum_{\substack{m \ge 1 \\ \lambda_m \le \lambda}} 1$$

denote the counting function of the λ_m's.

Weyl's classical asymptotic formula for $N(\lambda; \Omega)$,

$$N(\lambda; \Omega) \sim \frac{|\Omega|_2}{4\pi}\lambda, \quad \lambda \to \infty, \tag{1}$$

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is now known for arbitrary Ω, see [**1**], [**12**]. Weyl conjectured that if $\partial\Omega$ is sufficiently regular then there is a secondary term in (1) that is asymptotically a constant depending on the one-dimensional measure of $\partial\Omega$ times $\lambda^{1/2}$. Ivrii (see [**7**]) essentially proved this conjecture.

There remains the issue of when $\partial\Omega$ is not sufficiently regular, in particular if the boundary has a fractal dimension larger than 1. In 1979, Berry suggested a modified Weyl conjecture with a secondary term for $N(\lambda;\Omega)$ proportional to a constant times $\lambda^{d/2}$, where d is the Hausdorff dimension of $\partial\Omega$.

However, the Hausdorff measure of $\partial\Omega$ depends on the relative placement of the connected components of Ω in the ambient space, yet the eigenvalues do not care about this placement, so the Berry conjecture cannot strictly be true, see Lapidus [**8**]. Brossard and Carmona [**2**] had earlier demonstrated a specific counterexample to the Berry conjecture, and suggested instead that the Minkowski dimension D of $\partial\Omega$ is the more appropriate parameter. In fact, it was shown in [**8**] (also see [**9**]) that if $\partial\Omega$ has Minkowski dimension D with $1 < D < 2$ and finite upper Minkowski content in this dimension, then the error in (1) is $O(\lambda^{D/2})$. This led Lapidus [**8**] to conjecture that if in addition it was assumed that $\partial\Omega$ is Minkowski measurable in dimension D, then there would be a secondary term in (1) of the form $c\lambda^{D/2}$, with c a positive constant depending on the Minkowski content of $\partial\Omega$.

The analogue of this modified Weyl–Berry conjecture for regions in $\mathbf{R}^1$ was subsequently proved in [**10**], with a simplified proof given in [**5**]. However, for dimension 2 the conjecture is false in general. If a set of Newtonian capacity zero is removed from a given domain, the eigenvalues are not changed, yet the Minkowski dimension and content can be altered by strategically choosing which set of capacity zero to remove. This idea was developed in [**6**] and [**11**]. However, this behavior seems to be simple to bar with a further modification of the Weyl–Berry conjecture by stating it in terms of the "intrinsic" Minkowski dimension of the boundary, where we take the infimum of Minkowski dimensions of the boundaries of domains that agree up to a set of Newtonian capacity 0, and also the "intrinsic" Minkowski content, defined in the same way.

A more compelling counterexample was given in [**11**], involving sprays of the 1×1 unit square and of the 1×2 rectangle. (A "spray" is a disjoint union of similar copies of some given simple region with bounded total area.) However, the argument in [**11**] was not sufficient to give a counterexample for all Minkowski dimensions D with $1 < D < 2$, but rather for all but a possible countable set. In this note we show that the construction in [**11**] actually works for every D with $1 < D < 2$: there are no exceptions.

As in [**11**], the counterexample extends in a natural way to higher-dimensional ambient spaces $\mathbf{R}^n$ for $n \geq 2$.

Our argument involves getting precise formulations of the coefficient of the secondary terms for $N(\lambda;\Omega)$ for our two domains Ω in terms of the Riemann zeta-function and the Dedekind zeta-function for the quadratic field $\mathbf{Q}(i)$. We then show that these two functions are different for all D with $1 < D < 2$. Along the way we prove some results of perhaps independent interest about the zeta-functions involved. For example, we show that $(s-1)\zeta(s)$ is monotone for real $s \geq 1/2$. We show the same for $L(s,\chi)$, where χ is the quadratic Dirichlet character mod 4.

2. Our domains

Fix an arbitrary number D with $1 < D < 2$. Our first domain, denoted Ω_1, is the disjoint union of the $j^{-1/D} \times j^{-1/D}$ open squares for $j = 1, 2, \ldots$ arranged in the plane so that they don't overlap and sit inside a large disc. (It is routine to show that such an arrangement is possible; for a particularly efficient packing, see Moon and Moser [**13**].)

Let a be the positive real number $(2/(D+2))^{1/D}$ and let Ω_2' be the disjoint union of the $aj^{-1/D} \times 2aj^{-1/D}$ rectangles for $j = 1, 2, \ldots$ also arranged in the plane so that they don't overlap and sit inside a large disc.

The area of Ω_1 is $\zeta(2/D)$, where ζ is the Riemann zeta-function. The area of Ω_2' is $2a^2\zeta(2/D)$. Note that since $1 < D < 2$, we have $2a^2 < 1$, so that the area of Ω_2' is smaller than the area of Ω_1. Let Ω_2 be the disjoint union of Ω_2' and a square of area $(1-2a^2)\zeta(2/D)$. Thus, Ω_1 and Ω_2 have the same area.

As in [**11**], we have that the boundaries of Ω_1 and Ω_2 both have Minkowski dimension D with Minkowski content in dimension D of $2^{3-D}(2-D)^{-1}(D-1)^{-1}$.

Let $\zeta_1(s)$ be the spectral zeta-function for the 1×1 square and let $\zeta_2(s)$ be the spectral zeta-function for the $a \times 2a$ rectangle. Also let $N(\lambda;\Omega_i)$ denote the counting function of the eigenvalues for the Dirichlet Laplacian on Ω_i, for $i = 1, 2$. From [**11**, Theorem 3.2] we have

$$N(\lambda;\Omega_i) = \frac{1}{4\pi}\zeta(2/D)\lambda + (\zeta_i(D/2) + o(1))\lambda^{D/2}, \quad \lambda \to \infty, \tag{2}$$

for $i = 1, 2$. Note that the eigenvalues for the additional square tacked on to Ω_2' affect the main term for $N(\lambda;\Omega_2)$ (and is taken into account in (2)) and create an error of $O(\lambda^{1/2})$, which is negligible. That is, these eigenvalues are invisible to the secondary term.

The argument in [**11**] depended on ζ_1, ζ_2 being non-identical analytic functions, and so the secondary term coefficients in (2) could agree for at most countably many D in $(1, 2)$. Our goal in this paper is to show that they actually are unequal for all D in $(1, 2)$. Towards this end, we obtain explicit descriptions of the spectral zeta-functions ζ_1, ζ_2.

3. Our spectral zeta-functions

The eigenvalues for the 1×1 square are the numbers $\pi^2(m^2+n^2)$ where m, n run over positive integers. Thus,

$$\zeta_1(s) = \sum_{m,n>0} \frac{1}{\pi^{2s}(m^2+n^2)^s}.$$

This function resembles the Dedekind zeta-function for the Gaussian field $\mathbf{Q}(i)$, namely

$$\zeta_{\mathbf{Q}(i)}(s) = \sum_{\mathfrak{I}\neq 0} \frac{1}{N(\mathfrak{I})^s} = \frac{1}{4} \sum_{(m,n)\neq(0,0)} \frac{1}{(m^2+n^2)^s},$$

where $\mathfrak{I}$ runs over the nonzero ideals of $\mathbf{Z}[i]$. For a pair $m, n > 0$ that we see in $\zeta_1(s)$, there are 4 corresponding terms $(\pm m, \pm n)$ giving the same value to m^2+n^2, and this 4-fold appearance in the last sum is compensated by the $\frac{1}{4}$ in front of it. In addition, $\zeta_{\mathbf{Q}(i)}(s)$ has terms coming from pairs $(\pm m, 0)$ and $(0, \pm n)$ that have no counterpart in $\zeta_1(s)$. These extra terms contribute $\zeta(2s)$ to $\zeta_{\mathbf{Q}(i)}(s)$. Thus,

$$\pi^{2s}\zeta_1(s) = \zeta_{\mathbf{Q}(i)}(s) - \zeta(2s). \tag{3}$$

Let χ be the Dirichlet character mod 4; that is χ is defined on all integers n, with $\chi(n) = 0, 1, -1$ depending, respectively, on whether n is even, $n \equiv 1 \pmod 4$, $n \equiv -1 \pmod 4$. Consider the L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

We know that $L(s,\chi)$ is an entire function, and the series for it converges uniformly on compact subsets of $\Re s > 0$. It is of interest to us via the formula

$$\zeta_{\mathbf{Q}(i)}(s) = \zeta(s)L(s,\chi). \tag{4}$$

Now we look at $\zeta_2(s)$. We take a as in the last section. We have the eigenvalues $a^{-2}\pi^2(m^2+n^2/4)$, where $m, n > 0$ are integers. We find it more convenient to work with $4^{-s}a^{-2s}\pi^{2s}\zeta_2(s)$, giving us

$$4^{-s}a^{-2s}\pi^{2s}\zeta_2(s) = \sum_{m,n>0} \frac{1}{(4m^2+n^2)^s}.$$

Let $r_2(k)$ be the number of representations of k as $4m^2+n^2$ with $m, n > 0$. Further, let $r(k)$ denote the number of representations of k as m^2+n^2, where m, n are any integers. We have

$$\sum_{m,n>0} \frac{1}{(4m^2+n^2)^s} = \sum_{k=1}^{\infty} \frac{r_2(k)}{k^s}, \qquad \sum_{(m,n)\neq(0,0)} \frac{1}{(m^2+n^2)^s} = \sum_{k=1}^{\infty} \frac{r(k)}{k^s}.$$

Here are some observations on $r_2(k)$. First note that for $k \equiv 2 \pmod 4$, we have $r_2(k) = 0$, since squares are never $2 \pmod 4$. Next note that for

k odd, a representation of k as a sum of two squares must have one of the squares even and one odd. From this we see that

$$r_2(k) = \begin{cases} \frac{1}{8}r(k), & k \text{ not a square,} \\ \frac{1}{8}r(k) - \frac{1}{2}, & k \text{ is a square.} \end{cases}$$

Indeed, if $m, n > 0$ and $k = 4m^2 + n^2$ with n odd, then there are 8 corresponding representations of k as a sum of two squares, namely $(\pm 2m, \pm n), (\pm n, \pm 2m)$. The expression $\frac{1}{8}r(k)$ also counts an additional $\frac{1}{2}$ if k is a square, so the formula holds for odd k.

Now consider even values of k, so as we have seen, we may assume that $4 \mid k$. We claim in this case that we have

$$r_2(k) = \begin{cases} \frac{1}{4}r(k/4), & k \text{ not a square,} \\ \frac{1}{4}r(k/4) - 1, & k \text{ is a square.} \end{cases}$$

Indeed, a representation of k as $4m^2 + n^2$ has n even, so that $k/4 = m^2 + (n/2)^2$. Further, a pair m, n with $m, n > 0$ gives rise to 4 signed representations of $k/4$. In addition, there are 4 additional representations of $k/4$ as a sum of two squares when k is a square. So $\frac{1}{4}r(k)$ needs to be decreased by 1 in this case.

Putting these thoughts together, we have

$$\begin{aligned} 4^{-s}a^{-2s}\pi^{2s}\zeta_2(s) &= \sum_{k=1}^{\infty} \frac{r_2(k)}{k^s} \\ &= \sum_{\substack{k>0 \\ k \text{ odd}}} \left(\frac{r(k)/8}{k^s} - \frac{1/2}{k^{2s}} \right) + \sum_{\substack{k>0 \\ 4|k}} \left(\frac{r(k/4)/4}{k^s} - \frac{1}{(k/2)^{2s}} \right) \\ &= \frac{1}{2} \sum_{\substack{k>0 \\ k \text{ odd}}} \frac{r(k)/4}{k^s} + 4^{-s}\zeta_{\mathbf{Q}(i)}(s) - \frac{1}{2}\zeta(2s) - \frac{1}{2}2^{-2s}\zeta(2s). \end{aligned}$$

Now, $r(k)/4$ is multiplicative, and the local factor corresponding to the prime 2 in the Euler product for $\zeta_{\mathbf{Q}(i)}(s)$ is $(1 - 2^{-s})^{-1}$, so that

$$\frac{1}{2} \sum_{\substack{k>0 \\ k \text{ odd}}} \frac{r(k)/4}{k^s} = \frac{1}{2}(1 - 2^{-s})\zeta_{\mathbf{Q}(i)}(s).$$

Thus, with the above calculation, we have

$$4^{-s}a^{-2s}\pi^{2s}\zeta_2(s) = \left(\frac{1}{2} - 2^{-1-s} + 4^{-s} \right) \zeta_{\mathbf{Q}(i)}(s) - \frac{1}{2}(1 + 4^{-s})\zeta(2s),$$

so

$$a^{-2s}\pi^{2s}\zeta_2(s) = \left(2^{2s-1} - 2^{s-1} + 1\right) \zeta_{\mathbf{Q}(i)}(s) - \frac{1}{2}(4^s + 1)\zeta(2s).$$

We have proved the following result.

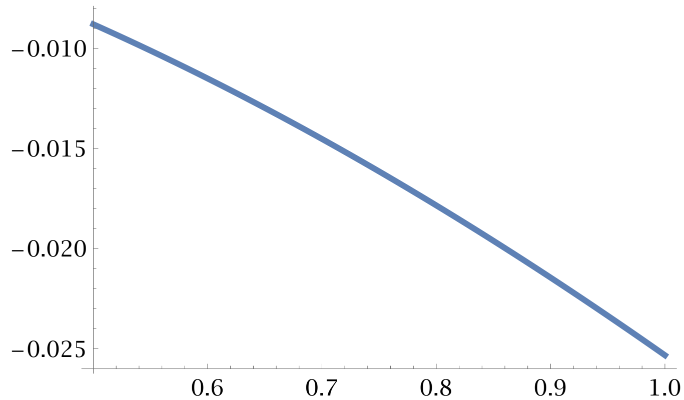

FIGURE 1. `Mathematica` plot of the right-hand side of (5) on $(1/2, 1)$.

PROPOSITION 3.1. *With the notation defined earlier, we have*

$$\pi^{2s}\zeta_1(s) = \zeta_{\mathbf{Q}(i)}(s) - \zeta(2s),$$
$$a^{-2s}\pi^{2s}\zeta_2(s) = (2^{2s-1} - 2^{s-1} + 1)\zeta_{\mathbf{Q}(i)}(s) - \frac{1}{2}(4^s + 1)\zeta(2s).$$

4. Are they equal?

Our task is to show that $\zeta_1(D/2) \neq \zeta_2(D/2)$ for $1 < D < 2$. With $s = D/2$ we have $a^{-2s} = D/2 + 1 = s + 1$. So, from Proposition 3.1, we would like to show that

(5)
$$(s+1)\pi^{2s}(\zeta_1(s)-\zeta_2(s)) = (s-2^{2s-1}+2^{s-1})\zeta_{\mathbf{Q}(i)}(s) - \left(s + \frac{1}{2} - 2^{2s-1}\right)\zeta(2s)$$

is nonzero for $\frac{1}{2} < s < 1$. In Figure 1 we present a `Mathematica` plot of the expression in (5), and though it is close to 0, one can plainly see that it is not 0. Is this a proof? Not quite, since there may conceivably be some wild gyrations of the functions between the discrete points used by `Mathematica` to form the plot. In this section we give the details necessary to prove that the expression in (5) is negative for s in $(1/2, 1)$.

We begin with the following result.

PROPOSITION 4.1. *The function $(s-1)\zeta(s)$ is increasing on $[1/2, \infty)$.*

PROOF. For $\Re s > 0, s \neq 1$, we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty x^{-1-s}\{x\}\,dx, \tag{6}$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. For $\Re s > 1$, this well-known formula follows from the definition of $\zeta(s)$ as a Dirichlet series and partial summation; for $\Re s > 0, s \neq 1$, it follows by analytic continuation. The same argument applied to

$$\zeta'(s) = \sum_{n=1}^{\infty} -n^{-s} \log n,$$

gives us

$$\zeta'(s) = \frac{-1}{(s-1)^2} - \int_1^{\infty} (-sx^{-1-s} \log x + x^{-1-s})\{x\}\, dx$$

for $\Re s > 0, s \neq 1$. Thus,

$$((s-1)\zeta(s))' = \zeta(s)+(s-1)\zeta'(s) = 1-\int_1^{\infty} (-s(s-1)x^{-1-s} \log x+(2s-1)x^{-1-s})\{x\}\, dx.$$

(This identity can also be obtained by differentiating $s-1$ times the equation in (6).) The integrand is positive for $s \in (1/2, 1)$, so replacing $\{x\}$ with 1 gives a lower bound on this interval. That is,

$$((s-1)\zeta(s))' > 1 - \int_1^{\infty} -s(s-1)x^{-1-s} \log x + (2s-1)x^{-1-s}\, dx = 0,$$

and the proposition is proved for the interval $[1/2, 1]$.

We now deal with the range $s \geq 1$. We have, as is easy to see,

$$(1-2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}.$$

Let $h(s)$ be the sum of the first 5 terms of this series, and let $h_k(s) = -k^{-s} + (k+1)^{-s}$, so that

$$(1-2^{1-s})\zeta(s) = h(s) + \sum_{k=3}^{\infty} h_{2k}(s).$$

It is easy to see that for $k \geq 3$, the function $h_k(s)$ is increasing for $s \geq 1$. (The derivative is $k^{-s} \log k - (k+1)^{-s} \log(k+1)$ and this is positive on $s \geq 1$ when $(k+1)/k > \log(k+1)/\log k$, which holds for $k \geq 3$.) We now show that $h(s)$ is also increasing for $s \geq 1$.

We have

$$3^s h'(s) = \left(\frac{3}{2}\right)^s \log 2 - \log 3 + \left(\frac{3}{4}\right)^s \log 4 - \left(\frac{3}{5}\right)^s \log 5.$$

Call this function $g(s)$. Then

$$g'(s) = \left(\frac{3}{2}\right)^s \log\frac{3}{2} \log 2 - \left(\frac{3}{4}\right)^s \log\frac{4}{3} \log 4 + \left(\frac{3}{5}\right)^s \log\frac{5}{3} \log 5.$$

It is easy to check that the sum of the first two terms here is positive for $s \geq 1$, it only involves checking that $(\log \frac{4}{3} \log 4)/(\log \frac{3}{2} \log 2) < 2$. So, $g'(s) > 0$ for $s \geq 1$, which in turn implies that $g(s)$ is increasing for $s \geq 1$. But $g(1) > 0$, so we have $g(s) > 0$ for $s \geq 1$, which in turn implies that $h(s)$ is increasing for $s \geq 1$. Thus, $(1-2^{1-s})\zeta(s)$ is increasing for $s \geq 1$.

It suffices now to show that $(s-1)/(1-2^{1-s})$ is increasing and positive for $s>1$. It is clearly positive. Letting $x=s-1$ and taking the derivative, we get

$$\frac{1-2^{-x}-x2^{-x}\log 2}{(1-2^{-x})^2}=\frac{2^x-1-x\log 2}{2^x(1-2^{-x})^2},$$

which is seen to be positive by using the Taylor expansion for 2^x in the numerator. This completes the proof. $\square$

We also need a monotonicity result for $L(s,\chi)$.

Proposition 4.2. *The function* $L(s,\chi)$ *is increasing on* $[1/2,\infty)$.

Proof. Write $L(s,\chi)=-f_1(s)+f_2(s)+f_3(s)$, where $f_1(s)=-(1+1/5^s+1/9^s+\cdots+1/49^s)$, $f_2(s)=-(1/3^s+1/7^s+\cdots+1/47^s)$, and

$$f_3(s)=\sum_{j=13}^{\infty}\left(\frac{-1}{(4j-1)^s}+\frac{1}{(4j+1)^s}\right).$$

Let $s\geq 1/2$. We have

$$f_3'(s)=\sum_{j=13}^{\infty}\left(\frac{\log(4j-1)}{(4j-1)^s}-\frac{\log(4j+1)}{(4j+1)^s}\right).$$

The function $x\mapsto(\log x)x^{-s}$ is decreasing for $x>e^{1/s}$, and thus for all $x\geq 8$. Hence, each summand in $f_3'(s)$ is positive, and so $f_3'(s)>0$. Therefore, the proposition will follow if $f_2'(s)-f_1'(s)>0$ for all $s\in[1/2,1]$.

We give a computer-assisted proof of this last inequality. Observe that

$$f_2'(s)=\log(3)/3^s+\log(7)/7^s+\cdots+\log(47)/47^s$$

while

$$f_1'(s)=\log(5)/5^s+\log(9)/9^s+\cdots+\log(49)/49^s.$$

In particular, both $f_1'(s)$ and $f_2'(s)$ are decreasing on $[1/2,1]$. To verify that $f_2'(s)-f_1'(s)>0$ on all of $[1/2,1]$, partition $[1/2,1]$ into $N:=10^4$ equal-length subintervals $[x_i,x_{i+1}]$ for $i=0,\dots,N-1$, where each $x_i=1/2+i/(2N)$. The minimum of $f_2'(s)-f_1'(s)$ on $[x_i,x_{i+1}]$ is bounded below by $f_2'(x_{i+1})-f_1'(x_i)$. Using `gp/pari`, one can easily check that $f_2'(x_{i+1})-f_1'(x_i)>0.004$ for all $i=0,\dots,N-1$.

This shows that $L(1,\chi)$ is increasing on $[1/2,1]$. To complete the proof, fix $s\geq 1$ and note that since $\log(x)/x^s$ is decreasing for $x\geq 3$, it follows that $L'(s,\chi)$ is positive. Hence $L(s,\chi)$ is increasing on $[1/2,\infty)$. $\square$

Remark. There is an alternative approach to Propositions 4.1 and 4.2 based on the Hadamard product decompositions of $\zeta(s)$ and $L(s,\chi)$. We discuss how this goes for $L(s,\chi)$ first, since the argument is slightly more involved than for $(s-1)\zeta(s)$.

We start from the formula for $\frac{L'}{L}$ found as equation (17) on p. 83 of [**4**]. This gives that for all real s,

$$\frac{L'(s,\chi)}{L(s,\chi)} - \frac{L'(0,\chi)}{L(0,\chi)} = \frac{1}{2}\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \frac{1}{2}\frac{\Gamma'(\frac{s}{2}+\frac{1}{2})}{\Gamma(\frac{s}{2}+\frac{1}{2})} + \sum_{\rho} \Re\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{7}$$

where ρ runs over the zeros of $L(s,\chi)$ in the critical strip $0 \leq \Re(s) \leq 1$. (If we did not take real parts in the last summand, then (7) would hold for all complex s.) When $s > 0$, we have $\Re(\frac{1}{s-\rho} + \frac{1}{\rho}) \geq 0$ as long as

$$\Re(\rho)(s - \Re(\rho)) + \Im(\rho)^2 > 0. \tag{8}$$

Restrict now to real $s > 0$. Then $\Re(\rho)(s - \Re(\rho)) \geq -1$. To get a handle on $\Im(\rho)$, we compare eq. (17) of [**4**, p. 83], taken at $s = 0$, with eq. (18) from the same page; this yields

$$\frac{L'(0,\chi)}{L(0,\chi)} + \frac{1}{2}\log\frac{4}{\pi} + \frac{1}{2}\frac{\Gamma'(1/2)}{\Gamma(1/2)} = -\sum_{\rho}\Re\frac{1}{\rho}.$$

From [**3**, Corollary 10.3.2, p. 188 and Proposition 10.3.5, pp. 189–190], we have $L'(0,\chi) = \log\frac{\Gamma(1/4)}{2\cdot\Gamma(3/4)}$ and $L(0,\chi) = \frac{1}{2}$. It follows that

$$\sum_{\rho}\Re\frac{1}{\rho} = 0.0777839\ldots. \tag{9}$$

Note that each term in this sum is nonnegative. We now take the subsum of (9) where $\Re\rho \geq \frac{1}{2}$. If ρ is any zero in the critical strip, then $1 - \overline{\rho}$ is also a zero with the same imaginary part. So our subsum consists of all zeros on the critical line and for each zero in the critical strip not on the critical line, we take the member of the pair $\rho, 1 - \overline{\rho}$ with the larger real part. Now

$$\Re\frac{1}{\overline{\rho}} + \Re\frac{1}{\rho} = \frac{2\cdot\Re\rho}{(\Re\rho)^2 + (\Im\rho)^2} \geq \frac{1}{1 + (\Im\rho)^2}.$$

Since ρ and $\overline{\rho}$ are both nontrivial zeros of $L(s,\chi)$, (9) implies that $|\Im\rho| > 3.4$. Thus, (8) holds for $s > 0$ (for every ρ). Consequently, the sum on ρ in (7) is nonnegative for these values of s. Turning to the digamma terms, recall that

$$-\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} + \gamma + \sum_{k=1}^{\infty}\left(\frac{1}{z+k} - \frac{1}{k}\right);$$

this follows, e.g., by logarithmically differentiating equation (2) on p. 73 of [**4**]. Hence, $-\frac{\Gamma'(z)}{\Gamma(z)}$ is a decreasing function of z for real $z > 0$. Therefore, when $0 < s \leq 1$,

$$\frac{1}{2}\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \frac{1}{2}\frac{\Gamma'(\frac{s}{2}+\frac{1}{2})}{\Gamma(\frac{s}{2}+\frac{1}{2})} \geq \frac{1}{2}\left(\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \frac{\Gamma'(1)}{\Gamma(1)}\right) = \log\frac{1}{2}.$$

(Here the final equality can be obtained from the partial fraction expansion of digamma given above.) Plugging back into (7), we deduce that for $0 <$

$s \le 1$,

$$\frac{L'(s,\chi)}{L(s,\chi)} \ge 2\log\frac{\Gamma(1/4)}{2\cdot\Gamma(3/4)} + \log\frac{1}{2}.$$

This last expression is larger than 0.09, and in particular is positive. Since $L(s,\chi) > 0$ for $s > 0$, it follows that $L'(s,\chi) > 0$ on $(0, 1]$. Thus, with the final step in the proof of Proposition 4.2 we have $L(s,\chi)$ increasing on $[0,\infty)$.

A similar method will show that $(s-1)\zeta(s)$ is increasing on $(0,\infty)$. Notice that for $s > 0$, we have $((s-1)\zeta(s))' > 0$ exactly when $\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} > 0$. Eq (7) on p. 80 of [**4**], combined with the expression for B on p. 81, shows that for all real s,

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = -\frac{1}{2}\gamma - 1 + \log(2\pi) - \frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} + \sum_\rho \Re\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

where ρ now runs over the nontrivial Riemann zeta zeros. As remarked on p. 82 of [**4**], $|\Im\rho| > 6$ for all ρ. (It is known in fact that $|\Im\rho| > 14$.) It follows from our earlier arguments that the the sum on ρ is nonnegative for all $s > 0$. Since $-\frac{\Gamma'(z)}{\Gamma(z)}$ is decreasing for real $z > 0$, we deduce that for $0 < s \le 4$,

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \ge -\frac{1}{2}\gamma - 1 + \log(2\pi) - \frac{1}{2}\frac{\Gamma'(3)}{\Gamma(3)} > 0.08.$$

It remains to show that $(s-1)\zeta(s)$ is increasing for $s > 4$. Using the idea at the end of the proof of Proposition 4.1, it suffices to show that $(1-2^{1-s})\zeta(s)$ is increasing in this range of s. Now $((1-2^{1-s})\zeta(s))' = \log(2)/2^s - \log(3)/3^s + \dots$. When $s > 4$, the terms $\log(x)/x^s$ are decreasing for $x \ge 2$. Hence, $((1-2^{1-s})\zeta(s))' > 0$.

Remark. Concerning specifically Proposition 4.1, Harold Diamond has shown us a proof by somewhat different methods that $(s-1)\zeta(s)$ is monotone for $s \ge -2.5$, which is nearly best possible.

Referring back to (5), we see that

$$(s+1)\pi^{2s}(\zeta_1(s)-\zeta_2(s)) = (s - 2^{2s-1} + 2^{s-1})\zeta_{\mathbf{Q}(i)}(s) - \left(s + \frac{1}{2} - 2^{2s-1}\right)\zeta(2s)$$

(10)

$$= \frac{2^{2s-1} - s - \frac{1}{2}}{s - \frac{1}{2}}\cdot\frac{1}{2}(2s-1)\zeta(2s) - \frac{(s - 2^{2s-1} + 2^{s-1})}{1-s}\cdot(s-1)\zeta(s)\cdot L(s,\chi).$$

Each of $\frac{1}{2}(2s-1)\zeta(2s)$, $(s-1)\zeta(s)$, and $L(s,\chi)$ is positive on $[1/2, 1]$, and our work above shows that these functions are increasing there. The next proposition supplies the corresponding results for the remaining factors in (10).

PROPOSITION 4.3. *Both of the functions*

$$\frac{s-2^{2s-1}+2^{s-1}}{1-s} \quad \text{and} \quad \frac{2^{2s-1}-s-\frac{1}{2}}{s-\frac{1}{2}}$$

are positive and increasing on $(0,\infty)$. *(We assume here that the discontinuities have been filled in to make the functions continuous.)*

PROOF. We first prove that both functions are increasing on the entire real line. We begin by recalling a fact from calculus about convex functions: *Suppose that* g *is a* $\mathcal{C}^2$ *function on an open interval* I. *For each* $x, y \in I$, *put*

$$S(x,y) = \begin{cases} \frac{g(x)-g(y)}{x-y} & \text{if } x \neq y, \\ g'(x) & \text{if } x = y. \end{cases}$$

If $g'' > 0$ *on* I, *then* $S(x,y)$ *is increasing separately in both* x, y. Applying this with $g(x) = 2^{2x-1} - x - 2^{x-1}$ and $x = 1, y = s$ shows that the first function is increasing. To handle the second function, take $g(x) = 2^{2x-1} - x - 1/2$, and look at $x = s, y = 1/2$. Since both functions vanish at 0, their positivity on $(0,\infty)$ is now immediate. We remark that this calculus fact could also have been used for the last step in the proof of Proposition 4.1. □

We can now prove our main result.

PROOF THAT $\zeta_1(s) \neq \zeta_2(s)$ FOR $1/2 < s < 1$. Let $F(s)$ denote the first term on the right-hand side of (10), and let $G(s)$ denote the second, subtracted term. Then F and G are positive, increasing functions on $[1/2, 1]$. We will prove that $F - G < 0$ on $[1/2, 1]$ by the same method employed in the proof of Proposition 4.2. We partition $[1/2, 1]$ into $N := 50$ equal length intervals $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, N-1$, with each $x_i = 1/2 + i/(2N)$. The maximum of $F - G$ on $[x_i, x_{i+1}]$ is at most $F(x_{i+1}) - G(x_i)$. Using `Mathematica`, one easily computes that each of these differences is smaller than -0.001. □

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