JOINT DISTRIBUTION IN RESIDUE CLASSES OF POLYNOMIAL-LIKE MULTIPLICATIVE FUNCTIONS

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Abstract. Under fairly general conditions, we show that families of integer-valued polynomial-like multiplicative functions are uniformly distributed in coprime residue classes mod p, where p is a growing prime (or nearly prime) modulus. This can be seen as complementary to work of Narkiewicz, who obtained comprehensive results for fixed moduli.

1. Introduction

For any integer-valued arithmetic function, it is reasonable to ask how the values of f are distributed in arithmetic progressions. As stated, this problem is far too general; to get any traction, it is necessary to restrict f. Let us suppose that f is multiplicative and that f is polynomial-like, in the sense that there is a polynomial $F(T) \in \mathbb{Z}[T]$ such that $f(p) = F(p)$ for every prime number p. In this case, Narkiewicz (beginning in [Nar67]) has made a comprehensive study of the distribution of f in coprime residue classes. For a thorough survey of this work, see Chapter V in [Nar84]. See also [Nar12] for a more recent contribution to this subject by the same author.

In 1982, Narkiewicz [Nar82] observed that his methods could be applied to study the joint distribution of several functions. We state a special case of the main theorem of [Nar82]. Let $f_1, \dots, f_K$ be a finite sequence of multiplicative, integer-valued arithmetic functions. Say that $f_1, \dots, f_K$ is nice if the following conditions hold:

(i) Each f_k is polynomial-like for a nonconstant polynomial: There is a nonconstant polynomial $F_k(T) \in \mathbb{Z}[T]$ such that $f_k(p) = F_k(p)$ for all primes p,

and

(ii) $F_1(T) \cdots F_K(T)$ has no multiple roots.

If $f_1, \dots, f_K$ is a nice family, a prime p is called good for $f_1, \dots, f_K$ if (a) $p > 5$, (b) $p > (1 + \sum_k \deg F_k(T))^2$, (c) p does not divide the leading coefficient of any $F_k(T)$, and (d) p does not divide the discriminant of

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$F_1(T)\cdots F_K(T)$. For any fixed nice family $f_1,\dots,f_K$, all but finitely many primes are good. Narkiewicz proves that if every prime divisor of q is good, and one restricts attention to n for which the values $f_1(n),\dots,f_K(n)$ are coprime to q, then those values are asymptotically jointly uniformly distributed among the coprime residue classes modulo q. More precisely: For every choice of integers $a_1,\dots,a_K$ coprime to q, we have

$$\sum_{\substack{n\le x\\ (\forall k)\ f_k(n)\equiv a_k \pmod q}} 1 \sim \frac{1}{\phi(q)^K}\sum_{\substack{n\le x\\ \gcd(\prod_{k=1}^K f_k(n),q)=1}} 1, \tag{1.1}$$

as $x\to\infty$. (It is proved along the way that the right-hand side of (1.1) tends to infinity under the same hypotheses.) In particular, we get joint uniform distribution in coprime residue classes mod p for all good primes p.

So far everything that has been said concerns the distribution to a fixed modulus q. It is natural to also consider the distribution when q grows with x. We prove a joint uniform distribution result of this kind for nice families valid when the modulus q is prime or "nearly prime". Here "nearly prime" means that $\delta(q)$ is small where

$$\delta(q) := \sum_{p\mid q}\frac{1}{p}.$$

Our main theorem is as follows.

Theorem 1.1. *Fix a nice sequence $f_1,\dots,f_K$ of multiplicative functions and fix $\epsilon>0$. Then (1.1) holds, uniformly as q, $x\to\infty$ with $\delta(q)=o(1)$ and $q\le(\log x)^{\frac{1}{K}-\epsilon}$, for every choice of coprime residue classes $a_1,\dots,a_K$* mod q.

In other words: For each $\eta>0$, there is a positive integer N (depending on $f_1,\dots,f_K$, ϵ, and η) such that the following holds. Suppose that $x>N$, that $(\log x)^{\frac{1}{K}-\epsilon}\ge q\ge N$, and that $\delta(q)<1/N$. Then for every K-tuple of integers $a_1,\dots,a_K$ coprime to q, the ratio of the LHS to the RHS in (1.1) *lies in $(1-\eta,1+\eta)$.*

For example, let $f_1(n)=n$, $f_2(n)=\phi(n)$, and $f_3(n)=\sigma(n)$. These form a nice family. By the result of Narkiewicz quoted above, the values of n, $\phi(n)$, $\sigma(n)$ coprime to p are uniformly distributed in coprime residue classes mod p for each fixed $p\ge 17$. It then follows from Theorem 1.1 that this equidistribution holds uniformly for $17\le p\le(\log x)^{\frac{1}{3}-\epsilon}$.

There are two directions in which one might hope to strengthen Theorem 1.1. First, it would be desirable to weaken the condition $\delta(q)=o(1)$, e.g., by replacing it with Narkiewicz's condition that q is divisible only by good primes. Such an improvement would seem to require a substantial new idea.

Second, one might hope to enlarge the range of allowable q past $(\log x)^{\frac{1}{K}-\epsilon}$. It was proved in [LLPSR] that when $K = 1$ and $f_1(n) = \phi(n)$, one can replace $(\log x)^{1-\epsilon}$ with $(\log x)^A$, for an arbitrary A, provided q is restricted to primes. This might seem to suggest that $(\log x)^{\frac{1}{K}-\epsilon}$ in Theorem 1.1 can always be replaced with $(\log x)^A$, with A arbitrary. As we now explain, this is too optimistic.

Suppose that $f_1, \dots, f_K$ is a fixed nice family with $K \geq 2$. Fix a prime p_0 with $f_1(p_0), \dots, f_K(p_0)$ all nonzero. Let $X := 2(\log x)^{\frac{1}{K-1}}$, and choose p to be a prime in $(2X/3, X]$. As $x \to \infty$, there are at "obviously" at least $(1 + o(1))x/p \log x \geq (\frac{4}{3} + o(1))x/p^K$ values of $n \leq x$ having $f_k(n) \equiv f_k(p_0) \pmod{p}$ for all $k = 1, \dots, K$, since n can be taken as any prime congruent to $p_0 \pmod{p}$. This shows that equidistribution in coprime residue cannot hold up to X. It is conceivable that in Theorem 1.1 uniformity holds up to $(\log x)^{\frac{1}{K-1}-\epsilon}$ (interpreted as $(\log x)^A$, A arbitrary, when $K = 1$). Again, it would seem to require a new idea to decide this.

We conclude this introduction with a brief summary of the proof of Theorem 1.1: Split off the first several largest prime factors of n, say $n = mP_J \cdots P_1$, where $P^+(m) \leq P_J \leq \cdots \leq P_1$. (Here J must be chosen judiciously; we also ignore n with fewer than J prime factors.) Most of the time, $P_J, \dots, P_1$ will appear to the first power only in n, so that $f_k(n) = f_k(m)f_k(P_J) \cdots f_k(P_1)$. Then given m, we use the prime number theorem for progressions (Siegel–Walfisz) and character sum estimates to understand the number of choices for $P_1, \dots, P_J$ compatible with the congruence conditions on $f_k(n)$.

Notation and conventions. Throughout, the letters p, P, r, with or without subscripts, always denote primes whether or not this is explicitly mentioned. We use $P^+(n)$ for the largest prime factor of n, with the convention that $P^+(1) = 1$. We write $\mathfrak{f}(\chi)$ for the conductor of the Dirichlet character χ.

2. Preparation

2.1. **Sieve lemmas.** We will make frequent use of the following special case of the fundamental lemma of sieve theory, as formulated in [HR74, Theorem 7.2, p. 209].

Lemma 2.1. *Let $X \geq Z \geq 3$. Suppose that the interval $\mathcal{I} = (u, v]$ has length $v - u = X$. Let $\mathcal{P}$ be a set of primes not exceeding Z. For each $p \in \mathcal{P}$, choose a residue class a_p mod p. The number of integers $n \in \mathcal{I}$ not congruent* *to* $a_p \bmod p$ *for any* $p \in \mathcal{P}$ *is*

$$X\left(\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)\right)\left(1+O\left(\exp\left(-\frac{1}{2}\frac{\log X}{\log Z}\right)\right)\right).$$

The following application of Lemma 2.1 yields a lower bound for the "numerator" on the right-hand side of (1.1). See Scourfield's Theorem 4 in [Sco84] for a closely related result (and compare with [Sco85]).

Lemma 2.2. *Fix a nice arithmetic function* f *(meaning that* f *is nice when viewed as a singleton sequence). Suppose that* $q, x \to \infty$ *with* $q = x^{o(1)}$ *and* $\delta(q) = o(1)$*. The number of* $n \leq x$ *for which* $\gcd(f(n), q) = 1$ *eventually*[1] *exceeds*

$$\frac{1}{20}x \prod_{\substack{p\leq x\\ \gcd(f(p),q)>1}}\left(1-\frac{1}{p}\right). \tag{2.1}$$

Remark.

(a) With a small amount of additional effort, one could show that (2.1) is the correct order of magnitude for this count of n. But we will not need this.

(b) It will be useful momentarily to know that the product on p in (2.1) has size at least $(\log x)^{o(1)}$. To see this, choose $F(T) \in \mathbb{Z}[T]$ with $f(p) = F(p)$ for all p. It suffices to show that

$$\sum_{\substack{p\leq x\\ \gcd(f(p),q)>1}} 1/p = o(\log\log x).$$

Let $\mathcal{S}$ be the set of primes $p \leq x$ with $\gcd(f(p), q) > 1$. For each prime r dividing q, let $\mathcal{S}_r = \{p \in (r, x] : F(p) \equiv 0 \pmod r\}$. Since F has $O_f(1)$ roots modulo every prime r,

$$\sum_{r\mid q}\sum_{p\in\mathcal{S}_r}\frac{1}{p} \ll_f \log\log x \sum_{r\mid q}\frac{1}{r} = \delta(q)\log\log x = o(\log\log x).$$

Here the sum on $p \in \mathcal{S}_r$ has been estimated by partial summation and the Brun–Titchmarsh inequality. For each r dividing q, there are $O_f(1)$ primes $p \leq r$ with $F(p) \equiv 0 \pmod r$. So if we put $\mathcal{S}' := \mathcal{S} \setminus \cup_{r\mid q}\mathcal{S}_r$, then $\#\mathcal{S}' \ll_f \omega(q)$, and, writing p_k for the kth prime in the usual increasing order,

$$\sum_{p\in\mathcal{S}'}\frac{1}{p} \leq \sum_{k=1}^{\#\mathcal{S}'}\frac{1}{p_k} \ll_f \log\log(3\omega(q)) = o(\log\log x),$$

using the simple bound $\omega(q) = O(\log x)$ in the last step.

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[1] meaning whenever q, x are sufficiently large and $\frac{\log q}{\log x}, \delta(q)$ are sufficiently small

Proof of Lemma 2.2. Fix a real number $U \geq 2$. We start by considering all $n \leq x$ not divisible by any $p \leq x^{1/U}$ with $\gcd(f(p), q) > 1$. For large q, x and small $\frac{\log q}{\log x}, \delta(q)$, where here and below "large" and "small" may depend on U, the sieve shows that the count of such n is

$$x\Bigg(\prod_{\substack{p \leq x^{1/U}\\ \gcd(f(p),q)>1}} \left(1 - \frac{1}{p}\right)\Bigg)(1 + O(\exp(-U/2))).$$

We now bound from above the number of these n with $\gcd(f(n), q) > 1$.

For each n surviving our initial sieve but having $\gcd(f(n), q) > 1$, we factor $n = A_1A_2B$, where

$$A_1 = \prod_{\substack{p \| n\\ \gcd(f(p),q)>1}} p, \quad A_2 = \prod_{\substack{p^e \| n,\ e>1\\ \gcd(f(p^e),q)>1}} p^e, \quad \text{and} \quad B = n/A_1A_2.$$

Then either $A_1 > 1$ or $A_2 > 1$. Moreover, every prime dividing A_1 exceeds $x^{1/U}$.

Suppose $A_2 > 1$. Since A_2 is squarefull, the number of $n \leq x$ with $A_2 > x^{1/2}$ is $O(x^{3/4})$, which will be negligible for our purposes. So we assume that $A_2 \leq x^{1/2}$. Given A_2, we count the number of possibilities for the cofactor A_1B. Note that $A_1B \leq x/A_2$ and that A_1B is free of prime factors $p \leq x^{1/U}$ with $\gcd(f(p), q) > 1$. So the sieve shows that the number of possibilities for A_1B is at most

$$\frac{x}{A_2}\Bigg(\prod_{\substack{p \leq x^{1/U}\\ \gcd(f(p),q)>1}} \left(1 - \frac{1}{p}\right)\Bigg)(1 + O(\exp(-U/4)))\,.$$

(We assume as usual that q, x are large and $\frac{\log q}{\log x}, \delta(q)$ are small.) Since

$$\sum_{M \text{ squarefull}} \frac{1}{M} = \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \dots\right) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = 1.943\dots,$$

the count of n with $A_2 > 1$ is bounded above by

$$0.945x\Bigg(\prod_{\substack{p \leq x^{1/U}\\ \gcd(f(p),q)>1}} \left(1 - \frac{1}{p}\right)\Bigg)(1 + O(\exp(-U/4)))\,.$$

Suppose now that $A_2 = 1$. Then $n = A_1B$, where $A_1 > 1$ and every prime dividing A_1 exceeds $x^{1/U}$. Let p be a prime dividing A_1, and write $A_1 = pS$. Then $n = pSB \leq x$ where $SB \leq x^{1-1/U}$. Given S and B, the number of possible p (and hence possible n) is, by Brun–Titchmarsh, at most

$$\sum_{r \mid q} \sum_{\substack{p \leq x/SB\\ F(p) \equiv 0 \pmod r}} 1 \ll_f \sum_{r \mid q} \frac{x}{rSB \log{(x/SBr)}} \ll \delta(q)U\frac{x}{\log x}\frac{1}{SB};$$

here we have assumed that $q \le x^{1/2U}$, so that $x/SBr \ge (x/SB)/r \ge x^{1/2U}$ for every $r \mid q$. Summing on S and B, the number of n that arise is

$$\ll_f \delta(q) U \frac{x}{\log x} \Bigg(\sum_{\substack{S \\ p \mid S \Rightarrow p \in (x^{1/U}, x]}} \frac{1}{S} \Bigg) \Bigg(\sum_{\substack{B \le x \\ p \mid B,\ p \le x^{1/U} \Rightarrow \gcd(f(p),q)=1}} \frac{1}{B} \Bigg)$$

$$\le \delta(q) U \frac{x}{\log x} \Bigg(\prod_{x^{1/U} < p \le x} \left(1 - \frac{1}{p}\right)^{-2} \Bigg) \Bigg(\prod_{\substack{p \le x^{1/U} \\ \gcd(f(p),q)=1}} \left(1 - \frac{1}{p}\right)^{-1} \Bigg),$$

which is

$$\ll \delta(q) U^3 \frac{x}{\log x} \prod_{p \le x^{1/U}} \left(1 - \frac{1}{p}\right)^{-1} \prod_{\substack{p \le x^{1/U} \\ \gcd(f(p),q)>1}} \left(1 - \frac{1}{p}\right)$$

$$\ll \delta(q) U^2 x \prod_{\substack{p \le x^{1/U} \\ \gcd(f(p),q)>1}} \left(1 - \frac{1}{p}\right).$$

But $\delta(q) = o(1)$, so the final expression is $o(x \prod_{p \le x^{1/U},\ \gcd(f(p),q)>1} (1 - 1/p))$.

Collecting estimates shows that if U is fixed sufficiently large, then eventually the number of $n \le x$ with $\gcd(f(n), q) = 1$ exceeds

$$\frac{1}{20} x \prod_{\substack{p \le x^{1/U} \\ \gcd(f(p),q)>1}} \left(1 - \frac{1}{p}\right).$$

Bounding the product over $p \le x^{1/U}$ below by the product over $p \le x$ completes the proof. $\square$

Our second application of the sieve is an upper bound on the count of n with few large prime factors. More precise results on this problem have been obtained by [Ten00], but the comparatively simple Lemma 2.3 below will suffice for our purposes.

Set $P_1^+(n) = P^+(n)$ and define, inductively,

$$P_{j+1}^+(n) = P^+(n/P_1^+(n) \cdots P_j^+(n)).$$

Thus, $P_j^+(n)$ is the jth largest prime factor of n (with multiple primes counted multiply), with $P_j^+(n) = 1$ if n has fewer than j prime factors.

Lemma 2.3. *Let $x \ge y \ge 10$. Let J be an integer, $J \ge 2$. The number of $n \le x$ with $P_J^+(n) \le y$ is*

$$\ll_J x \frac{\log y}{\log x} (\log\log x)^{J-1}.$$

Proof. Suppose that $P_J^+(n) \le y$ and write $n = AB$, where A is the largest divisor of n composed of primes not exceeding y. Then $\omega(B) \le \Omega(B) < J$.

Clearly, $A \le x^{1/2}$ or $B \le x^{1/2}$. Suppose first that $A \le x^{1/2}$. Then $B \le x/A$ and $\omega(B) \le J-1$, so that by a classical theorem of Landau (see [HW08, Theorem 437, p. 491]), given A there are $\ll \frac{x}{A\log(x/A)}(\log\log(x/A))^{J-2} \ll \frac{x}{A\log x}(\log\log x)^{J-2}$ possible B. Summing $1/A$ on A with $P^+(A) \le y$ introduces a factor $\prod_{p\le y}(1-1/p)^{-1} \ll \log y$, which yields for this case a slightly stronger upper bound than that claimed in the lemma.

Suppose now that $B \le x^{1/2}$. Since A has no prime factors larger than y, the sieve shows that given B, the number of possible $A \le x/B$ is $\ll \frac{x}{B}\prod_{y<p\le x^{1/2}}(1-1/p) \ll \frac{x}{B}\frac{\log y}{\log x}$. Since

$$\sum_{\substack{B\le x\\ \omega(B)\le J-1}} \frac{1}{B} \le \sum_{j=0}^{J-1}\frac{1}{j!}\left(\sum_{p^e\le x}\frac{1}{p^e}\right)^j \ll_J (\log\log x)^{J-1},$$

the result follows. □

2.2. **Character sums of polynomials.**

We require estimates for (complete, multiplicative) character sums of polynomials modulo prime powers. For prime moduli, we use the following version of the Weil bound.

Lemma 2.4. *Let $\mathbb{F}_q$ be a finite field, and let $\chi_1, \dots, \chi_K$ be characters of $\mathbb{F}_q^\times$, extended to all of $\mathbb{F}_q$ by setting $\chi_k(0) = 0$. Let $F_1(T), \dots, F_K(T) \in \mathbb{F}_q[T]$ be nonzero and pairwise relatively prime. Assume that for some $1 \le k \le K$, the polynomial $F_k(T)$ is not an $\operatorname{ord}(\chi_k)$th power in $\mathbb{F}_q[T]$ or a constant multiple of such. Then*

$$\left|\sum_{x\in\mathbb{F}_q}\chi_1(F_1(x))\cdots\chi_K(F_K(x))\right| \le \Big(\sum_{k=1}^{K} d_k - 1\Big)\sqrt{q},$$

where d_k denotes the degree of the largest squarefree divisor of $F_k(T)$.

Lemma 2.4 is essentially Corollary 2.3 of [Wan97]. It is assumed in [Wan97] that all the χ_k are nontrivial, but this assumption is not used in the proof.

Estimating the sums to proper prime power moduli requires some stage setting. Let p^m be an odd prime power, where $m \ge 2$. Let g be a primitive root modulo p^m. Let χ be the Dirichlet character mod p^m defined on integers x coprime to p by

$$\chi(x) = \exp\left(2\pi \mathrm{i}\frac{\operatorname{ind}_g(x)}{p^{m-1}(p-1)}\right), \tag{2.2}$$

where $g^{\operatorname{ind}_g(x)} \equiv x \pmod{p^m}$.

Let $F(T) \in \mathbb{Z}[T]$ be a nonconstant polynomial, and let t be the largest nonnegative integer for which p^t divides every coefficient of $F'(T)$. Let $\tilde{F}(T) \in \mathbb{F}_p[T]$ denote the mod p reduction of $p^{-t}F'(T)$. (Note that $\tilde{F}(T)$ is nonzero by the choice of t.) Let $\mathcal{A} \subset \mathbb{F}_p$ denote the set of roots of $\tilde{F}(T)$ in $\mathbb{F}_p$ that are not roots of the reduction of $F(T)$ mod p. For each $\alpha \in \mathcal{A}$, let ν_α denote the multiplicity of α as a zero of $\tilde{F}(T)$, and let $M = \max_{\alpha\in\mathcal{A}} \nu_\alpha$.

The following is an immediate consequence of Cochrane's Theorem 1.2 in [Coc02]; that very general result concerns mixed additive and multiplicative character sums, but see Theorem 2.1 of [CLZ03] for the specialization to multiplicative character sums.

Lemma 2.5. *Under the above conditions, and the additional assumption that $m \geq t+2$, we have*

$$\left|\sum_{x \bmod p^m} \chi(F(x))\right| \leq \Big(\sum_{\alpha\in\mathcal{A}} \nu_\alpha\Big) p^{\frac{t}{M+1}} p^{m(1-\frac{1}{M+1})}.$$

The proof of Theorem 1.1 depends on the following consequence of Lemmas 2.4 and 2.5, which seems of some independent interest.

Proposition 2.6. *Let $F_1(T), \dots, F_K(T) \in \mathbb{Z}[T]$ be nonconstant and assume that the product $F_1(T)\cdots F_K(T)$ has no multiple roots. Let p be an odd prime not dividing the leading coefficient of any of the $F_k(T)$ and not dividing the discriminant of $F_1(T)\cdots F_K(T)$. Let m be a positive integer, and let $\chi_1, \dots, \chi_K$ be Dirichlet characters modulo p^m, at least one of which is primitive. Then*

$$\left|\sum_{x \bmod p^m} \chi_1(F_1(x))\cdots\chi_K(F_K(x))\right| \leq (D-1)p^{m(1-1/D)}, \tag{2.3}$$

where $D = \sum_{k=1}^K \deg F_k(T)$.

Proof. Take first the case when $m=1$. When $D=1$, the left-hand side of (2.3) vanishes and (2.3) holds. When $D \geq 2$, we apply Lemma 2.4 with $q=p$. The mod p reductions of the $F_k(T)$ are nonzero (in fact, of the same degree as their counterparts in $\mathbb{Z}[T]$), and $F_1(T)\cdots F_K(T)$ is squarefree over $\mathbb{F}_p$, so that each $F_k(T)$ is squarefree and the $F_k(T)$ are pairwise relatively prime in $\mathbb{F}_p[T]$. Since some χ_k is primitive, it has order larger than 1, and so $F_k(T)$ is not an $\mathrm{ord}(\chi_k)$th power in $\mathbb{F}_q[T]$ or a constant multiple of such. Lemma 2.4 now yields (2.3).

Henceforth, we suppose that $m \geq 2$. Let g be a primitive root mod p^m, and let χ be the character mod p^m defined in (2.2). We can write each χ_k in the form χ^{A_k}, where $0 < A_k \le p^{m-1}(p-1)$. Then

$$\sum_{x \bmod p^m} \chi_1(F_1(x)) \cdots \chi_K(F_K(x)) = \sum_{x \bmod p^m} \chi(F(x)), \tag{2.4}$$

where

$$F(T) := F_1(T)^{A_1} \cdots F_K(T)^{A_K}.$$

Also,

$$F'(T) = \left(\prod_{k=1}^{K} F_k(T)^{A_k - 1}\right) G(T),$$

$$\text{where} \quad G(T) := \sum_{k=1}^{K} \left(A_k F_k'(T) \prod_{\substack{1 \le j \le K \\ j \ne k}} F_j(T) \right).$$

Let t be the largest integer for which p^t divides all the coefficients of $F'(T)$. Since none of the $F_k(T)$ are multiples of p, the power p^t is also the largest power of p dividing all the coefficients of $G(T)$ (by Gauss's content lemma).

We claim that $t = 0$. Choose, for each $k = 1, \ldots, K$, a root α_k of $F_k(T)$ from the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Then in $\overline{\mathbb{F}}_p$,

$$G(\alpha_k) = (F_k'(\alpha_k) \prod_{\substack{1 \le j \le K \\ j \ne k}} F_j(\alpha_k)) A_k,$$

and the factor in front of A_k is nonzero. But if $t > 0$, then $G(T)$ induces the zero function on $\overline{\mathbb{F}}_p$, forcing each A_k to be a multiple of p. Then none of the χ_k are primitive characters mod p^m, contrary to hypothesis.

Now let $\mathcal{A}$, ν_α, and M be defined as in the discussion preceding Lemma 2.5. Then each $\alpha \in \mathcal{A}$ is a root in $\mathbb{F}_p$ of the mod p reduction of $G(T)$ of multiplicity ν_α. Moreover, $M \le \sum_{\alpha \in \mathcal{A}} \nu_\alpha \le \deg G(T) \le D - 1$. The desired upper bound (2.3) follows from (2.4) and Lemma 2.5. $\square$

3. Proof of Theorem 1.1

Throughout this proof, we suppress the dependence of implied constants or implied lower/upper bounds on the constant $\epsilon > 0$ as well as the family $f_1, \ldots, f_K$. We let $F_1(T), \ldots, F_K(T) \in \mathbb{Z}[T]$ be such that $f_k(p) = F_k(p)$ for all primes p. We put

$$J := (K+1)D$$

where, anticipating an application of Proposition 2.6,

$$D := 1 + \sum_{k=1}^{K} \deg F_k(T).$$

It will be convenient to introduce the notation

$$\sum\nolimits_{\mathbf{f}}(x;q) := \sum_{\substack{n\le x\\ \gcd(f(n),q)=1}} 1.$$

Throughout this proof, when we say a term is ignorable, we mean that it is of smaller order than the right-hand side of (1.1), that is, $o(\phi(q)^{-K}\sum_{\mathbf{f}}(x;q))$.

By Lemma 2.2 (with $f = f_1\cdots f_K$) and the remark following it, we find that

$$\begin{aligned}\phi(q)^{-K}\sum\nolimits_{\mathbf{f}}(x;q) &\ge q^{-K}x(\log x)^{o(1)}\\ &\ge x(\log x)^{K\epsilon+o(1)}/\log x \ge x(\log x)^{\epsilon+o(1)}/\log x.\end{aligned}$$

(Here we use our assumption that $q \le (\log x)^{\frac{1}{K}-\epsilon}$.) So Lemma 2.3 allows us to discard from the left-hand side of (1.1) those n for which $P_J^+(n) \le L$, where

$$L := \exp((\log x)^{\frac{1}{2}\epsilon}),$$

at the cost an ignorable error. Write each remaining n in the form $n = mP_J\cdots P_1$, where each $P_j = P_j^+(n)$. We keep only those n where $P^+(m) < P_J < \cdots < P_1$. Any n discarded at this step has a repeated prime factor exceeding L, and there are $O(x/L)$ of these, which is again ignorable. Note that for all of the remaining n, we have $f(n) = f(m)f(P_J)\cdots f(P_1)$, where each $P_j > L_m$ with

$$L_m := \max\{P^+(m), L\}.$$

By the observations of the last paragraph, it suffices to prove that

$$\sum\nolimits_{\mathbf{f}}(x;q,\mathbf{a}) \sim \frac{1}{\phi(q)^K}\sum\nolimits_{\mathbf{f}}(x;q),$$

where

$$\begin{aligned}\sum\nolimits_{\mathbf{f}}(x;q,\mathbf{a}) :&= \sum_{\substack{m\le x\\ \gcd(\prod_{k=1}^K f_k(m),q)=1}} \sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ L_m<P_J<\cdots<P_1\\ (\forall k)\ f_k(m)\prod_{j=1}^J f_k(P_j)\equiv a_k \pmod q}} 1\\
&= \sum_{\substack{m\le x\\ \gcd(\prod_{k=1}^K f_k(m),q)=1}} \frac{1}{J!} \sum_{\substack{P_1,\dots,P_J \text{ distinct}\\ P_1\cdots P_J\le x/m\\ \text{each } P_j > L_m\\ (\forall k)\ f_k(m)\prod_{j=1}^J f_k(P_j)\equiv a_k \pmod q}} 1. \qquad (3.1)\end{aligned}$$

We now remove the distinctness restriction in the final inner sum. Estimating crudely, this incurs an error of size $O(x/mL)$ in the inner sum and an error of size $O(x\log x/L)$ in the double sum.

For each $k = 1, 2, \dots, K$, let u_k denote a value of $f_k(m)^{-1}a_k$ mod q and define

$$V_m := \{(v_1 \bmod q, \dots, v_J \bmod q) : \gcd(v_1 \cdots v_J, q) = 1, \qquad (\forall k) \prod_{j=1}^{J} F_k(v_j) \equiv u_k \pmod q\}.$$

Then writing $\mathbf{v} = (v_1 \bmod q, \dots, v_j \bmod q)$,

$$\sum_{\substack{P_1,\dots,P_J \\ P_1\cdots P_J \le x/m \\ \text{each } P_j > L_m \\ (\forall k)\ f_k(m)\prod_{j=1}^{J} f_k(P_j) \equiv a_k \pmod q}} 1 = \sum_{\mathbf{v}\in V_m} \sum_{\substack{P_1,\dots,P_J \\ P_1\cdots P_J \le x/m \\ \text{each } P_j > L_m \\ (\forall j)\ P_j \equiv v_j \pmod q}} 1.$$

For each $\mathbf{v} \in V_m$, we show how to remove the right-hand congruence conditions on the P_j. First we handle P_1. Noting that $q \le (\log x) = (\log L)^{2/\epsilon}$, the Siegel–Walfisz theorem (see, for example, [MV07, Corollary 11.21]) implies that for a certain positive constant $C = C_\epsilon$,

$$\sum_{\substack{P_1,\dots,P_J \\ P_1\cdots P_J \le x/m \\ \text{each } P_j > L_m \\ (\forall j)\ P_j \equiv v_j \pmod q}} 1 = \sum_{\substack{P_2,\dots,P_J \\ P_2\cdots P_J \le x/m \\ \text{each } P_j > L_m \\ (\forall j\ge 2)\ P_j \equiv v_j \pmod q}} \sum_{\substack{L_m < P_1 \le \frac{x}{mP_2\cdots P_J} \\ P_1 \equiv v_1 \pmod q}} 1,$$

where

$$\sum_{\substack{L_m < P_1 \le \frac{x}{mP_2\cdots P_J} \\ P_1 \equiv v_1 \pmod q}} 1 = \frac{1}{\phi(q)} \sum_{L_m < P_1 \le \frac{x}{mP_2\cdots P_J}} 1 + O\left(\frac{x}{mP_2\cdots P_J} \exp(-C\sqrt{\log L})\right).$$

It follows that

$$\sum_{\substack{P_1,\dots,P_J \\ P_1\cdots P_J \le x/m \\ \text{each } P_j > L_m \\ (\forall j)\ P_j \equiv v_j \pmod q}} 1 = \frac{1}{\phi(q)} \sum_{\substack{P_1,P_2,\dots,P_J \\ P_1\cdots P_J \le x/m \\ \text{each } P_j > L_m \\ (\forall j\ge 2)\ P_j \equiv v_j \pmod q}} 1 + O\left(\frac{x}{m} \exp\left(-\frac{1}{2}C\sqrt{\log L}\right)\right).$$

In the same way, the congruence conditions on $P_2, \dots, P_J$ can be removed successively to yield

$$\sum_{\substack{P_1,\dots,P_J \\ P_1\cdots P_J \le x/m \\ \text{each } P_j > L_m \\ (\forall j)\ P_j \equiv v_j \pmod q}} 1 = \frac{1}{\phi(q)^J} \sum_{\substack{P_1,P_2,\dots,P_J \\ P_1\cdots P_J \le x/m \\ \text{each } P_j > L_m}} 1 + O\left(\frac{x}{m} \exp\left(-\frac{1}{2}C\sqrt{\log L}\right)\right).$$

The main term on the right-hand side is independent of $\mathbf{v}$. Keeping in mind that $\#V_m \le q^J \le (\log x)^J$ for all m, we deduce from (3.1) that

$$\sum\nolimits_{\mathbf{f}}(x;q,\mathbf{a}) = \sum_{\substack{m\le x\\ \gcd(\prod_{k=1}^K f_k(m),q)=1}} \frac{\#V_m}{\phi(q)^J}\cdot\frac{1}{J!}\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m}} 1 + O\left(x\exp\left(-\frac{1}{4}C\sqrt{\log L}\right)\right). \tag{3.2}$$

To handle the main term, notice that

$$\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m\\ \text{some } \gcd(f(P_j),q)>1}} 1 \le J\sum_{p\mid q}\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m\\ p\mid f(P_1)}} 1.$$

The condition that $p \mid f(P_1)$ puts P_1 in a certain (possibly empty) set of $O(1)$ residue classes mod p. Removing these congruence condition by the Siegel–Walfisz theorem (exactly as above) we find that (with C as above)

$$\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m\\ p\mid f(P_1)}} 1 \ll \frac{1}{p}\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m}} 1 + \frac{x}{m}\exp\left(-\frac{1}{2}C\sqrt{\log L}\right)$$

and so

$$\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m\\ \text{some } \gcd(f(P_j),q)>1}} 1 \ll \delta(q)\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m}} 1 + \frac{x}{m}\exp\left(-\frac{1}{4}C\sqrt{\log L}\right).$$

Since $\delta(q)=o(1)$,

$$\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m}} 1 = (1+O(\delta(q)))\sum_{\substack{P_1,\dots,P_J\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m,\ \gcd(f(P_j),q)=1}} 1 + O\left(\frac{x}{m}\exp\left(-\frac{1}{4}C\sqrt{\log L}\right)\right),$$

which (considering possible orderings of $P_1,\dots,P_J$) in turn is equal to

$$(1+O(\delta(q)))J!\sum_{\substack{P_J<\cdots<P_1\\ P_1\cdots P_J\le x/m\\ \text{each } P_j>L_m,\ \gcd(f(P_j),q)=1}} 1 + O\left(\frac{x}{m}\exp\left(-\frac{1}{4}C\sqrt{\log L}\right)\right).$$

The following claim will be established at the end of this section as an application of Proposition 2.6.

Claim. $\#V_m \sim q^J/\phi(q)^K$, uniformly in m.

We insert the estimate of the Claim, together with the last display, into (3.2). Since $\delta(q)=o(1)$, we have $\frac{q^J}{\phi(q)^J}(1+O(\delta(q)))=1+o(1)$. We find that up to an ignorable error, $\sum_{\mathbf{f}}(x;q,\mathbf{a})$ is equal to

$$(1+o(1))\frac{1}{\phi(q)^K}\sum_{\substack{m\le x\\ \gcd(\prod_{k=1}^K f_k(m),q)=1}}\quad\sum_{\substack{L_m<P_J<\cdots<P_1\\ P_1\cdots P_J\le x/m\\ \text{each }\gcd(f(P_j),\ q)\,=\,1}} 1.$$

We can view the double sum as counting those numbers $n\le x$ with $\gcd(f(n),q)=1$ and certain extra constraints: Namely, the Jth largest prime factor of n exceeds L and none of the largest J prime factors are repeated. But (by reasoning seen at the start of this proof) dropping the extra constraints incurs an ignorable error. So up to an ignorable error, $\sum_{\mathbf{f}}(x;q,\mathbf{a})$ is equal to $\frac{(1+o(1))}{\phi(q)^K}\sum_{\mathbf{f}}(x;q)$. By definition of ignorable,

$$\sum\nolimits_{\mathbf{f}}(x;q,\mathbf{a})\sim\frac{1}{\phi(q)^K}\sum\nolimits_{\mathbf{f}}(x;q),$$

and we have seen already that this suffices to complete the proof of Theorem 1.1.

Proof of the Claim. Using χ_0 for the trivial character mod q, orthogonality yields

$$\begin{aligned}
&\phi(q)^K\#V_m\\
&\quad=\sum_{\chi_1,\dots,\chi_K \bmod q}\Big(\prod_{k=1}^K\bar\chi_k(u_k)\Big)\Big(\sum_{x_1,\dots,x_J \bmod q}\chi_0(\prod_{j=1}^J x_j)\cdot\prod_{k=1}^K\chi_k(\prod_{j=1}^J F_k(x_j))\Big)\\
&\quad=\sum_{\chi_1,\dots,\chi_K \bmod q}\Big(\prod_{k=1}^K\bar\chi_k(u_k)\Big)S_{\chi_1,\dots,\chi_K}^J,
\end{aligned}\tag{3.3}$$

where

$$S_{\chi_1,\dots,\chi_K}:=\sum_{x \bmod q}\chi_0(x)\chi_1(F_1(x))\cdots\chi_K(F_K(x)).$$

The number of x mod q where one of $x,F_1(x),\dots,F_K(x)$ has a common factor with q is $\ll q\delta(q)=o(q)$, and so the tuple $\chi_1,\dots\chi_K$ of trivial characters makes a contribution $\sim q^J$ to (3.3). So to complete the proof, it suffices to show that

$$\sum_{\substack{\chi_1,\dots,\chi_K \bmod q\\ \text{not all trivial}}}|S_{\chi_1,\dots,\chi_K}|^J\tag{3.4}$$

has size $o(q^J)$.

Assume that $\chi_1, \dots, \chi_K$ are Dirichlet characters mod q, not all of which are trivial. Factor $q = \prod_{p \mid q} p^{e_p}$. Each character χ_k, for $k = 0, 1, \dots, K$, admits a unique decomposition of the form $\chi_k = \prod_{p \mid q} \chi_{k,p}$, where $\chi_{k,p}$ is a Dirichlet character modulo p^{e_p}. By the type of the tuple $\chi_1, \dots, \chi_K$, we mean the $\omega(q)$-element sequence of positive integers $\{\mathfrak{f}_p\}_{p \mid q}$, where each

$$\mathfrak{f}_p = \operatorname{lcm}[\mathfrak{f}(\chi_{1,p}), \dots, \mathfrak{f}(\chi_{K,p})].$$

Write $q = q_0 q_1$, where q_1 is the unitary divisor of q supported on the primes $p \mid q$ for which $\mathfrak{f}_p > 1$. Note that $q_1 > 1$, since not all of $\chi_1, \dots, \chi_K$ are trivial. By the Chinese remainder theorem,

$$S_{\chi_1, \dots, \chi_K} = \prod_{p \mid q} \Bigg(\sum_{x \bmod p^{e_p}} \chi_{0,p}(x) \chi_{1,p}(F_1(x)) \cdots \chi_{K,p}(F_K(x)) \Bigg),$$

from which we see that

$$\begin{aligned} |S_{\chi_1, \dots, \chi_K}| &\le q_0 \prod_{p \mid q_1} \Bigg| \sum_{x \bmod p^{e_p}} \chi_{0,p}(x) \chi_{1,p}(F_1(x)) \cdots \chi_{K,p}(F_K(x)) \Bigg| \\ &= q_0 \prod_{p \mid q_1} \frac{p^{e_p}}{\mathfrak{f}_p} \Bigg| \sum_{x \bmod \mathfrak{f}_p} \chi_{0,p}(x) \chi_{1,p}(F_1(x)) \cdots \chi_{K,p}(F_K(x)) \Bigg|. \end{aligned}$$

At least one of $\chi_{1,p}, \dots, \chi_{K,p}$ has conductor $\mathfrak{f}_p$, and so the remaining sum on x may be estimated by Proposition 2.6, yielding

$$|S_{\chi_1, \dots, \chi_K}| \le q (D-1)^{\omega(q_1)} \prod_{p \mid q_1} \mathfrak{f}_p^{-1/D}.$$

(If none of the $F_k(T)$ are multiples of T, we apply Proposition 2.6 with the polynomials $T, F_1(T), \dots, F_k(T)$; otherwise, the sum on x is unchanged if we remove the term $\chi_{0,p}(x)$ and we apply the proposition with $F_1(T), \dots, F_k(T)$. Keep in mind that since $\delta(q) = o(1)$, all the prime factors of q are large, so the nondivisibility conditions on p in Proposition 2.6 are certainly satisfied.) Hence (since $J = (K+1)D$) $|S_{\chi_1, \dots, \chi_K}|^J \le q^J (D-1)^{\omega(q_1) J} \prod_{p \mid q_1} \mathfrak{f}_p^{-(K+1)}$. There are no more than $(\prod_{p \mid q_1} \mathfrak{f}_p)^K$ tuples $\chi_1, \dots, \chi_K$ sharing this type, so that the contribution from all such tuples to (3.4) is at most $q^J (D-1)^{\omega(q_1) J} \prod_{p \mid q_1} \mathfrak{f}_p^{-1}$. Summing $\mathfrak{f}_p$ over all powers of p, for $p \mid q_1$, reveals that the contribution from all types corresponding to a given q_1 is at most

$$q^J (D-1)^{\omega(q_1) J} \frac{q_1}{\phi(q_1)} \prod_{p \mid q_1} p^{-1} \le q^J (D-1)^{\omega(q_1) J} 2^{\omega(q_1)} \prod_{p \mid q_1} p^{-1}.$$

Finally, summing over all unitary divisors q_1 of q with $q_1 > 1$ bounds (3.4) by

$$q^J \left(\prod_{p \mid q} \left(1 + \frac{2(D-1)^J}{p} \right) - 1 \right) \le q^J (\exp(2(D-1)^J \delta(q)) - 1) = o(q^J).$$

Collecting estimates completes the proof of the Claim. □

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