EULER AND THE PARTIAL SUMS OF THE PRIME HARMONIC SERIES

PAUL POLLACK

Abstract. In a 1737 paper, Euler gave the first proof that the sum of the reciprocals of the prime numbers diverges. That paper can be considered as the founding document of analytic number theory, and its key innovation — so-called Euler products — are now ubiquitous in the field. In this note, we probe Euler's claim there that "the sum of the reciprocals of the prime numbers" is "as the logarithm" of the sum of the harmonic series. Euler's argument for this assertion falls far short of modern standards of rigor. Here we show how to arrange his ideas to prove the more precise claim that

$$\Big| \sum_{\text{primes } p \leq x} \frac{1}{p} - \log\log x \Big| < 6$$

for all $x > \mathrm{e}^4$.

1. Introduction

Analytic number theory is one area of mathematics whose birthday can be specified with pinpoint accuracy: April 25, 1737. On that date, Euler presented a paper titled *Variae observationes circa series infinitas* (*Various observations about infinite series*) to the St. Petersburg Academy [2]. Of the many theorems in this paper, undoubtedly the most famous is the following seminal result.

Euler's Theorem 19. *The sum of the reciprocals of the prime numbers,*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \dots$$

is infinitely great but is infinitely times less than the sum of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots.$$

And the sum of the former is as the logarithm of the sum of the latter.

To a modern reader, Euler's handling of infinite quantities in this statement is puzzling. What does he mean when he writes that one infinite quantity is "infinitely many times less" than another? The claim that former is "as the logarithm" of the latter, far from helping to clarify matters, only contributes to the confusion.

One attempt to make sense of Euler's claims brings in the idea of partial sums. Euler knew well that $\sum_{n \leq x} \frac{1}{n} \approx \log x$ (indeed, his eponymous constant γ measures the limiting error in this approximation [1]). So when Euler claims that "the sum of the reciprocals of the prime numbers" is "as the logarithm" of the sum of the

2000 *Mathematics Subject Classification.* Primary: 11A41, Secondary: 11N05.

harmonic series, perhaps he is suggesting that for large values of x,

$$\sum_{p\le x}\frac{1}{p}\approx \log\log x. \tag{1}$$

Here and below, p always denotes a prime variable. If (1) is what Euler meant (as hypothesized by Sandifer [9, Chap. 33, p. 194], Weil [14, Chap. 3, p. 266], and others), then he was indeed correct. In 1874, Mertens [5] showed that the difference between the left and right-hand sides of (1) tends, as $x\to\infty$, to the finite limit

$$\gamma-\sum_{p}\sum_{k\ge 2}\frac{1}{kp^k}=0.2614972128\dots.$$

However, Mertens was working more than a century after Euler, and his methods were very different. For instance, Mertens' argument depends crucially on a result of Legendre [4, pp. 8–10] describing how $n!$ decomposes as a product of primes. Abel's method of partial summation also makes an appearance. Both of these innovations date to the early 19th century. So what could Euler, working in 1737, actually have known about the sum of prime reciprocals?

We review Euler's argument in §2 below. On the face of it, his proof — while sufficient to establish the divergence of the prime harmonic series — does not give any quantitative result in the direction of (1). It is natural to wonder whether Euler could have proved a version of (1) with the methods at his disposal. The object of this note is to give a simple proof, inspired by Euler, of the following estimate.

Theorem 1. *For all $x>\mathrm{e}^4$, we have*

$$\left|\sum_{p\le x}\frac{1}{p}-\log\log x\right|<6.$$

Theorem 1 sharpens a result of Pétermann [7], who gave a proof by "Eulerian" methods of the same inequality with 6 replaced by $\log\log\log x+C$ for a certain constant C. An estimate of roughly the same quality as Pétermann's, proved by a different elementary method, was obtained earlier by Treiber [13].

2. Euler's proof of Theorem 19

For real $s>1$, put $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ and $P(s)=\sum_p\frac{1}{p^s}$. It is usual to refer to $\zeta(s)$ as the *Euler–Riemann zeta function* and to $P(s)$ as the *prime zeta function.* The following proposition and its proof represent a modernized version of Euler's argument for his Theorem 19.

Proposition 2. *For all real $s>1$, we have* $0<\log\zeta(s)-P(s)<\frac{1}{2}$.

Proof. We use the famous "Euler factorization" of the Riemann zeta function (see [2, Theorem 8]). According to this result, we have for all $s>1$ that

$$\zeta(s)=\prod_p\frac{1}{1-\frac{1}{p^s}}.$$

(Euler only considers integral s, but his argument goes through without any changes for real $s>1$.) We now take the natural logarithm of both sides. Recalling that

$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$ for $|x| < 1$, we see that

$$\begin{aligned}\log\zeta(s) &= -\sum_p \log(1-p^{-s}) \\ &= P(s) + \sum_p \sum_{k\geq 2} \frac{1}{kp^{ks}}. \end{aligned} \tag{2}$$

Since $s > 1$, we have

$$0 < \sum_p \sum_{k\geq 2} \frac{1}{kp^{ks}} \leq \frac{1}{2}\sum_p \sum_{k\geq 2} \frac{1}{p^k} = \frac{1}{2}\sum_p \frac{1}{p(p-1)} < \frac{1}{2}\sum_{n\geq 2}\frac{1}{n(n-1)} = \frac{1}{2},$$

which with (2) gives the desired estimate. □

To obtain Theorem 19, Euler throws caution to the wind and sets $s = 1$ in Proposition 2. His conclusion is that the sum of the reciprocals of the primes, which is $P(1)$, differs by a bounded amount from the logarithm of the sum of the harmonic series, which is $\log\zeta(1)$.

It is not hard to turn Euler's proof into a rigorous demonstration that the sum of the reciprocals of the primes diverges. Indeed, suppose for a contradiction that $P(1)$ converges. Then the above argument shows that $\zeta(s)$ has a finite limit as $s \downarrow 1$, contradicting the divergence of $\zeta(1)$.

On the other hand, it seems clear that Euler's proof does not yield a quantitative form of (1) in any obvious way. Euler's (amended) argument gives us information about limiting behavior as $s \downarrow 1$, while to make (1) precise requires knowing about limiting behavior as $x \to \infty$. To have any hope of proving (1), a bridge needs to be built between these two worlds.

3. Proof of Theorem 1

To prove Theorem 1, we supplement Proposition 2 with the following simple bounds for $\zeta(s)$.

Lemma 3. *For all $s > 1$, we have $1 < (s-1)\zeta(s) < s$.*

Proof. Since t^{-s} is strictly decreasing for $t \geq 1$, we see that

$$(n+1)^{-s} < \int_n^{n+1} t^{-s}\,\mathrm{d}t < n^{-s}$$

for every positive integer n. Summing on n gives $\zeta(s)-1 < \int_1^\infty t^{-s}\,\mathrm{d}t = \frac{1}{s-1} < \zeta(s)$. Hence,

$$\frac{1}{s-1} < \zeta(s) < \frac{1}{s-1} + 1 = \frac{s}{s-1}.$$

Multiplying through by $s-1$ completes the proof. □

Combining Proposition 2 and Lemma 3 yields the following key estimate.

Lemma 4. *For all real $s \in (0, \frac{1}{2})$,*

$$\left|P(s+1) - \log\frac{1}{s}\right| < \frac{1}{2}.$$

Proof. Proposition 2 shows that $-\frac{1}{2} < P(s+1) - \log\zeta(s+1) < 0$. On the other hand, Lemma 3 shows that $1 < s\zeta(s+1) < \frac{3}{2}$, so that $0 < \log\zeta(s+1) - \log\frac{1}{s} < \log\frac{3}{2}$. Adding these inequalities, and using that $\log\frac{3}{2} < (\frac{3}{2}-1) = \frac{1}{2}$, completes the proof. $\square$

Proof of Theorem 1. We assume throughout that $x > \mathrm{e}^4$. If λ is a bounded function defined on the interval $[0,1]$, we set

$$\Sigma(\lambda; x) = \sum_p \frac{1/p}{p^{1/\log x}} \cdot \lambda(p^{-1/\log x}).$$

Notice that for the function

$$\lambda_0(t) := \begin{cases} 1/t & \text{if } 1/\mathrm{e} \le t \le 1, \\ 0 & \text{if } 0 \le t < 1/\mathrm{e}, \end{cases}$$

we have

$$\Sigma(\lambda_0; x) = \sum_{p \le x} \frac{1}{p}.$$

The idea of the proof is to replace λ_0 by linear polynomials λ which dominate it from above and below. Let $\lambda(t) = \ell_0 + \ell_1 t$. Then

$$\Sigma(\lambda; x) = \ell_0 \cdot P\left(1 + \frac{1}{\log x}\right) + \ell_1 \cdot P\left(1 + \frac{2}{\log x}\right).$$

Since $x > \mathrm{e}^4$, we have $\frac{2}{\log x} < \frac{1}{2}$; so from Lemma 4,

$$|\Sigma(\lambda; x) - \ell_0 \log\log x - \ell_1 \log\frac{\log x}{2}| \le \frac{|\ell_0|}{2} + \frac{|\ell_1|}{2}.$$

Writing $\log\frac{\log x}{2} = \log\log x - \log 2$ and noting that $\ell_0 + \ell_1 = \lambda(1)$ gives

$$|\Sigma(\lambda; x) - \lambda(1)\log\log x| \le \frac{|\ell_0|}{2} + |\ell_1|\left(\frac{1}{2} + \log 2\right). \tag{3}$$

We now prove Theorem 1 by making specific choices for λ, illustrated in Figure 1.

- **Upper bound**: Take $\lambda(t) = \lambda^{(U)}(t) := -\mathrm{e}t + (\mathrm{e}+1)$, so that the line $(t, \lambda(t))$ passes through $(1/\mathrm{e}, \mathrm{e})$ and $(1,1)$. Since the graph of $1/t$ is concave up on $[1/\mathrm{e}, 1]$, it follows that $\lambda^{(U)}(t) \ge \lambda_0(t)$ when $1/\mathrm{e} \le t \le 1$. Since $\lambda^{(U)}(t) > \mathrm{e} > 0$ for $0 \le t < 1/\mathrm{e}$, we also have $\lambda^{(U)}(t) \ge \lambda_0(t)$ in that range. So from (3),

$$\begin{aligned}\sum_{p\le x}\frac{1}{p} = \Sigma(\lambda_0; x) \le \Sigma(\lambda^{(U)}; x) &\le \log\log x + \frac{\mathrm{e}+1}{2} + \mathrm{e}\left(\frac{1}{2} + \log 2\right) \\ &< \log\log x + 6.\end{aligned}$$

- **Lower bound**: Take $\lambda(t) = \lambda^{(L)}(t) := \frac{\mathrm{e}}{\mathrm{e}-1}t - \frac{1}{\mathrm{e}-1}$, so that the line $(t, \lambda(t))$ passes through $(1/\mathrm{e}, 0)$ and $(1,1)$. Since $\lambda^{(L)}(t) < 0$ when $0 \le t < 1/\mathrm{e}$ and $\lambda^{(L)}(t) \le 1$ when $1/\mathrm{e} \le t \le 1$, we see that $\lambda^{(L)}(t) \le \lambda_0(t)$ for all $t \in [0,1]$. So from (3) again,

$$\begin{aligned}\sum_{p\le x}\frac{1}{p} \ge \Sigma(\lambda^{(L)}; x) &\ge \log\log x - \frac{1}{2(\mathrm{e}-1)} - \frac{\mathrm{e}}{\mathrm{e}-1}\left(\frac{1}{2} + \log 2\right) \\ &> \log\log x - 3.\end{aligned}$$

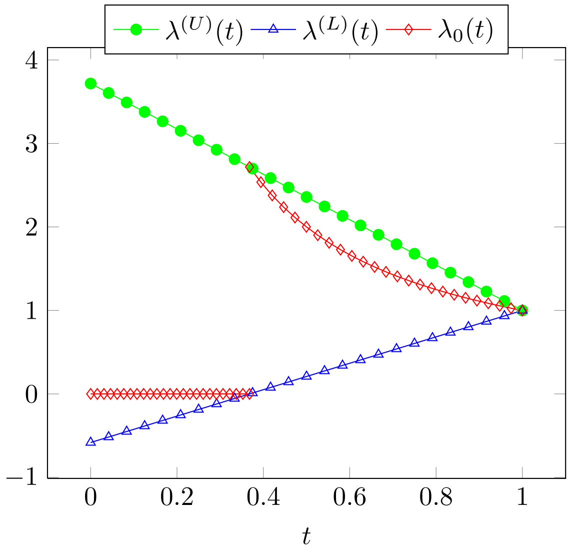

FIGURE 1. Graphs of the functions $\lambda^{(U)}$, $\lambda^{(L)}$, and λ_0 on $[0, 1]$.

This completes the proof of Theorem 1. □

Remarks.

(i) Being more careful about error terms, one can show that for any linear polynomial $\lambda(t) = \ell_0 + \ell_1 t$ satisfying $\lambda(1) = 1$, we have

$$\Sigma(\lambda; x) = \log\log x - \ell_1 \log 2 - C + o(1),$$

where $C := \sum_p \sum_{k\geq 2} \frac{1}{kp^k} = 0.3157184520\ldots$ and $o(1)$ denotes a quantity that tends to zero as $x \to \infty$. Choosing $\lambda^{(U)}$ and $\lambda^{(L)}$ as before, we now find that the constant 6 in Theorem 1 can be replaced with 2, provided that x is assumed large enough. We have preferred to write the proof to optimize readability rather than the final numerical result. In view of Mertens' later definitive work, numerical nitpicking seems pointless.

(ii) The chief novelty here is the upper bound. Indeed, it was observed already by Sylvester in 1888 [10] (and perhaps by others earlier) that

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1} = \sum_{n:\ p|n \Rightarrow p\leq x} \frac{1}{n} \geq \sum_{n\leq x}\frac{1}{n} \geq \log x$$

whenever $x > 1$. Now mimicking Euler's argument for Theorem 19 gives

$$\sum_{p\leq x}\frac{1}{p} > \log\log x - C,$$

where C is as in (i). (See [7, eq. (5)] and cf. [13, Satz 1].) This is superior to our lower bound. In our defense, we find it appealing to deduce both upper and lower estimates by a uniform method.

4. Putting our proof in its place

The argument of the last section can be viewed as an elementary piece of *Tauberian* reasoning. Roughly speaking, a *Tauberian theorem* is a device for converting asymptotic information about weighted sums into asymptotic information valid when the weights have been removed or replaced. The first result in this direction was proved by Tauber in 1897 [11]: *Suppose that* $\sum_{n=0}^{\infty} a_n z^n \to A$ *as* $z \uparrow 1$, *and that* $na_n \to 0$ *as* $n \to \infty$. *Then* $\sum_{n=0}^{\infty} a_n = A$.

In many applications to number theory, the weights to be stripped off are not of the form z^n but instead of the form n^{-s}; in other words, they come from Dirichlet series, not power series. Making obvious changes to the proof of Theorem 1, we arrive at the following simple Tauberian result for Dirichlet series with logarithmic singularities.

Proposition 5. *Suppose that the function* F *is given for real* $s > 1$ *by a convergent Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

with $a_n \geq 0$ *for all* n. *Suppose that as* $s \downarrow 1$, *the difference*

$$F(s) - \log \frac{1}{s-1}$$

remains bounded. Then as $x \to \infty$, *the difference*

$$\sum_{n \leq x} \frac{a_n}{n} - \log\log x$$

also remains bounded.

Theorem 1 corresponds to the case $F(s) = \log \zeta(s)$.

More sophisticated Tauberian theorems imply finer results about the distribution of primes. In fact, Tauberian theory furnishes what is arguably the simplest known approach to the prime number theorem. See, for example, the remarkably pithy expository article of Zagier [15], which is based on work of Newman [6] and Korevaar [3]. For further discussion of the role of Tauberian theorems in analytic number theory, the reader is invited to consult the comprehensive monographs of Postnikov (see especially [8, Chapter 1]) and Tenenbaum [12, Chapter II.7].

Acknowledgements. This note arose out of discussions with Carl Pomerance concerning Euler's proof of Theorem 19. Thanks are also due to Dominic Klyve, who read an early draft and made a number of valuable suggestions leading to improvements in the exposition. Finally, the author expresses his gratitude to the founders of the Euler Archive — Dominic Klyve, Lee Stemkoski, and Erik Tou — for making Euler's collected works freely available; see `http://www.eulerarchive.org`.

References

[1] L. Euler, *De progressionibus harmonicis observationes* (E43), Commentarii academiae scientiarum Petropolitanae **7** (1740), 150–161. Reprinted in *Opera omnia*, ser. I, vol. 14, pp. 87–100. Original article available online at `http://eulerarchive.maa.org/pages/E043.html`.

[2] ———, *Variae observationes circa series infinitas* (E72), Commentarii academiae scientiarum Petropolitanae **9** (1744), 160–188. Reprinted in *Opera omnia*, ser. I, vol. 14, pp. 217–244. Original article available online, along with an English translation by Pelegrí Viader Sr., Lluis Bibiloni, and Pelegrí Viader Jr., at `http://eulerarchive.maa.org/pages/E072.html`.

[3] J. Korevaar, *On Newman's quick way to the prime number theorem*, Math. Intelligencer **4** (1982), 108–115.

[4] A. M. Legendre, *Essai sur la théorie des nombres*, second ed., Courcier, Paris, 1808.

[5] F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie*, J. Reine Angew. Math. **78** (1874), 46–62.

[6] D. J. Newman, *Simple analytic proof of the prime number theorem*, Amer. Math. Monthly **87** (1980), 693–696.

[7] Y.-F. S. Pétermann, *Une preuve de* $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\&c.= l.l\infty$ *inspirée par Euler*, Elem. Math. **62** (2007), 167–173.

[8] A. G. Postnikov, *Introduction to analytic number theory*, Translations of Mathematical Monographs, vol. 68, American Mathematical Society, Providence, RI, 1988.

[9] C. E. Sandifer, *How Euler did it*, MAA Spectrum, Mathematical Association of America, Washington, DC, 2007.

[10] J. J. Sylvester, *On certain inequalities relating to prime numbers*, Nature **38** (1888), 259–262.

[11] A. Tauber, *Ein Satz aus der Theorie der unendlichen Reihen*, Monatsh. Math. **8** (1897), 273–277.

[12] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics, vol. 46, Cambridge University Press, Cambridge, 1995.

[13] D. Treiber, *Zur Reihe der Primzahlreziproken*, Elem. Math. **50** (1995), 164–166.

[14] A. Weil, *Number theory: An approach through history from Hammurapi to Legendre*, Modern Birkhäuser Classics, Birkhäuser, Boston, 2007.

[15] D. Zagier, *Newman's short proof of the prime number theorem*, Amer. Math. Monthly **104** (1997), 705–708.

Department of Mathematics, University of Georgia, Boyd Graduate Studies Research Center, Athens, Georgia 30602, USA

E-mail address: `pollack@uga.edu`