Big doings with small gaps

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AIM
Bounded gaps between primes

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The fundamental theorems of ZMT-ology

Theorem (Zhang for $m = 2$, Maynard–Tao for $m > 2$)

For each integer $m \geq 2$, there is a finite number $k_0(m)$ with the following property: Let $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ be an admisible k-tuple with $k \geq k_0(m)$. Here admissible means that

$$\#\{n \bmod p : \prod_{i=1}^{k}(n + h_i) \equiv 0 \pmod{p}\} < p$$

for every prime p. There are infinitely many n for which the list $n + h_1, \ldots, n + h_k$ contains at least m prime numbers.

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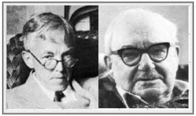
Conjecture (Hardy–Littlewood)

Can take $k_0(m) = m$.



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For each integer $m \geq 2$, there is a finite number $k_0(m)$ with the following property: Let $a_1 n + b_1, \ldots, a_k n + b_k$ be an admissible colletion of linear polynomials with $k \geq k_0(m)$. Here admissible means that

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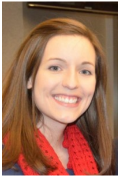
Conjecture (Dickson's *k*-tuples conjecture)

Can take $k_0(m) = m$.

B–F–T-B





Theorem (Banks–Freiberg–Turnage-Butterbaugh)

Suppose in the last theorem that all the a_i coincide, say each $a_i = A$. Assume the sequence $b_1, \ldots, b_k$ is monotonic. There are infinitely many n for which the list $An + b_1, \ldots, An + b_k$ contains at least m ***consecutive*** *prime numbers.*

Main idea of the proof.

Introduce extra congruence conditions on n forcing $An + b$ composite for each b between b_1 and b_k not among the b_i.

Example

Suppose we really did know that every pair of admissible linear forms assumed simultaneous prime values.

And say we wanted n and $n + 6$ to be consecutive primes.

Replace n with $15n + 1$: we apply the result to $15n + 1$ and $15n + 7$.

Proof and consequences

For each k, put $d_k = p_{k+1} - p_k$.

Conjecture (Erdős and Turán, 1948)

The sequence $\{d_k\}$ contains arbitrarily long (strictly) increasing runs and arbitrarily long (strictly) decreasing runs.

Proof (BFTB).

Let's treat the increasing case first. Given m, let $k = k_0(m)$, and apply BFTB to the collection $n + 2, n + 2^2, \ldots, n + 2^k$. Let's check admissibility.

Checking admissibility: If $p \neq 2$, then

$$\prod_{i=1}^{k}(n+2^i)\Big|_{n=0} \not\equiv 0 \pmod{p}.$$

whereas if $p = 2$, then

$$\prod_{i=1}^{k}(n+2^i)\Big|_{n=1} \not\equiv 0 \pmod{p}.$$

Thus, the list $n+2, \dots, n+2^k$ contains at least m consecutive primes. The sequence of gaps between them is increasing.

The decreasing case is similar, with the theorem applied to $n-2, \dots, n-2^k$.

Open problem: Show that there are infinitely many runs of consecutive prime gaps in the order `LOW HIGH LOW`. In other words, $d_k < d_{k+1}$ but $d_{k+1} > d_{k+2}$.

(If I remember correctly...) C. Spiro has shown this would follow if there is at least one $m \geq 4$ with $k_0(m) < 2^m$.

Shiu strings

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Each coprime residue class a mod q contains arbitrarily long runs of consecutive primes.

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Theorem (BFTB)

Shiu's theorem is still true fourteen years later. Moreover, it remains true even if one restricts the primes to lie in a bounded length interval. ("Bounded" means bounded in terms of q and the length of the run.)

For the proof, again let m be given, and let $k = k_0(m)$.
We apply the BFTB theorem to a collection of the form

$$qn + a_1, \ldots, qn + a_k$$

where each $a_i \equiv a \pmod{q}$.

Why is there an admissible collection like this?

Choose each $a_i \equiv a \pmod{q}$. If p is an obstruction to admissibility, then considering $n = 0$, we get $p \mid a_1 \cdots a_k$.

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Consequence: We get admissibility if we choose each $a_i \equiv a \pmod{q}$ to have no prime factors $\leq k$.

Some questions of Sierpiński

Let $s(n)$ denote the sum of the decimal digits of n. For example, $s(2014) = 2 + 1 + 4 = 7$. We can observe that

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Question (Sierpiński, 1961)

Given m, are there infinitely many m-tuples of consecutive primes $p_n, \ldots, p_{n+m-1}$ with

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Answer (Thompson and P.): Yes.

We sketch the proof. We let $k = k_0(m)$. We seek an admissible collection of the form

$$10^\ell n + b_1, \quad 10^\ell n + b_2, \quad \dots, \quad 10^\ell n + b_k,$$

where $0 < b_1 < b_2 < \cdots < b_k < 10^\ell$ and $s(b_1) = \cdots = s(b_k) = s$, say.

Given such a collection, BFTB says we get at least m consecutive primes, each of which has digit sum $s(n) + s$.

How do we ensure admissibility?
If p is an obstruction to admissibility, then $p \mid b_1 \cdots b_k$.
Moreover, either $p \mid 10$ or $p \leq k$.

Consequence: We get admissibility if we choose each b_i coprime to 10 and all primes $p \leq k$.

Can we choose distinct positive integers $b_1, \ldots, b_k$ coprime to $10\prod_{p\le k} p$ and all possessing the same digit sum?

Yes, by a direct elementary argument.

OR: Using a 2009 result of Mauduit and Rivat, one can actually pick the b_i to be primes. Their result shows there are "many" ℓ-digit primes p with $s(p) = s$, for all integers s "near" the expected mean sum-of-digits $\frac{9}{2}\ell$. (More precisely, they prove a "local central limit theorem" for sums of digits of primes.)

Another question of Erdős

Let $\sigma(\cdot)$ be the usual sum-of-divisors function, so $\sigma(n) = \sum_{d|n} d$.

Question

If $\sigma(a) = \sigma(b)$, what can be said about the ratio a/b?

Example

$\sigma^{-1}(8960) = \{3348, 5116, 5187, 6021, 7189, 7657\}$.

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Conjecture (Erdős, 1959)

Nothing. More precisely, the set of ratios $\{a/b : \sigma(a) = \sigma(b)\}$ *is dense in* $\mathbb{R}_{>0}$.

Theorem (P.)

Erdős's conjecture is true.

In this talk we focus on a special case.

Theorem

For every B, there is a pair of integers a and b with $\sigma(a) = \sigma(b)$ *and* $a/b > B$.

The proof uses ideas of Schinzel, who proved this special case assuming Dickson's conjecture.

Proof.

Let $k = k_0(2)$.
Notice that the ratio $\sigma(m)/m$ gets arbitrarily large as m ranges over the natural numbers, since

$$\sigma(m)/m = \sum_{d|m} \frac{1}{d},$$

and the harmonic series diverges.

Now choose integers $a_1, \ldots, a_k$ where each

$$\begin{aligned} \sigma(a_1)/a_1 &> B, \\ \sigma(a_2)/a_2 &> B \cdot \sigma(a_1)/a_1, \\ &\vdots \\ \sigma(a_k)/a_k &> B \cdot \sigma(a_{k-1})/a_{k-1}. \end{aligned}$$

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&\vdots \\
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\end{aligned}$$

Consider the admissible collection $\sigma(a_1)n - 1, \ldots, \sigma(a_k)n - 1$.
For infinitely many n, at least two of $\sigma(a_1)n - 1, \ldots, \sigma(a_k)n - 1$ are prime, say

$$p_i = \sigma(a_i)n - 1 \quad \text{and} \quad p_j = \sigma(a_j)n - 1.$$

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Notice $\sigma(p_i a_j) = \sigma(a_i)\sigma(a_j)n = \sigma(p_j a_i)$.
The ratio

$$\frac{p_j a_i}{p_i a_j} = \frac{p_j}{p_i} \cdot \frac{a_i}{a_j} = \frac{\sigma(a_j)n - 1}{\sigma(a_i)n - 1} \cdot \frac{a_i}{a_j}$$
$$> \frac{\frac{1}{2}\sigma(a_j)n}{\sigma(a_i)n} \cdot \frac{a_i}{a_j} 1 = \frac{1}{2}\frac{\sigma(a_j)/a_j}{\sigma(a_i)/a_i} \geq \frac{B}{2}.$$

Bounded gaps between primes in special sets

Say a set of primes $q_1, q_2, \ldots$ has the **bounded gaps property** if $\liminf_{n\to\infty} q_{n+m} - q_n < \infty$, for every m.

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Chebotarev sets have the bounded gaps property.

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Key input provided by an analogue of Bombieri–Vinogradov proved by Murty–Murty.

Theorem (Baker–Zhao)

Fix real numbers α *and* β *with* $\alpha > 1$ *and* α *irrational. Then the set of primes of the form* $\lfloor \alpha n + \beta \rfloor$ *has the bounded gaps property.*

cf. earlier work of Benatar and Chua–Park–Smith

Artin's primitive root conjecture

Conjecture (Artin, 1927)

Fix g not a square and $\neq -1$. *There are infinitely many primes p for which g is a primitive root mod p.*

Theorem (Hooley, 1967)

GRH for Dedekind zeta functions implies Artin's conjecture.

Theorem (P.)

Assume GRH for Dedekind zeta functions. The set of primes p with g as a primitive root has the bounded gaps property.

Sketch of Hooley's proof

For simplicity, we consider only $g = 2$.
We look for such primes $p \leq N$. Let $W = 4\prod_{p \leq D_0} p$, where $D_0 = \log\log\log N$.

First, we hit the problem with the W-trick:
Fix $p \equiv \nu \mod W$, so that $p \equiv 3 \pmod 8$ (so 2 is **not** a square mod p) and $p - 1$ has no odd prime factors $\leq D_0$.

There are $\approx \pi(N)/\phi(W)$ such $p \leq N$.

If 2 is not a primitive root mod p, then for some prime ℓ,

$$p \equiv 1 \pmod{\ell} \quad \text{and} \quad 2^{\frac{p-1}{\ell}} \equiv 1 \pmod{p}. \qquad (P_\ell)$$

From 1., we must have $\ell > D_0$. Consider three ranges of remaining ℓ:

$$D_0 < \ell < N^{1/2}/\log^3 N$$
$$N^{1/2}/\log^3 N \leq \ell < N^{1/2}\log^3 N$$
$$\ell \geq N^{1/2}\log^3 N.$$

We will show that the number of p possessing P_ℓ for ℓ in each of these three ranges is $o(\pi(N)/\phi(W))$.

Range I: $D_0 < \ell < N^{1/2}/\log^3 N$
Reinterpret P_ℓ as a splitting condition: it says p splits completely in $\mathbb{Q}(\zeta_\ell, \sqrt[\ell]{2})$. By GRH Chebotarev, the number of such $p \leq N$ is

$$\frac{1}{\ell(\ell-1)}\pi(N) + O(N^{1/2}\log N).$$

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Summming over ℓ gives a bound

$$\ll \frac{\pi(N)}{D_0} + N/(\log N)^2 = o(\pi(N)/\phi(W)).$$

Range II: $N^{1/2}/\log^3 N \le \ell < N^{1/2}\log^3 N$

From P_ℓ, keep only the condition that $p \equiv 1 \pmod{\ell}$.
By Brun–Titchmarsh, the number of such $p \le N$ is

$$\ll \frac{\pi(N)}{\ell}.$$

Summing on ℓ in our range gives

$$\ll \pi(N) \cdot \frac{\log\log N}{\log N},$$

which is $o(\pi(N)/\phi(W))$.

Range III: $\ell \geq N^{1/2} \log^3 N$

P_ℓ implies that p divides $2^j - 1$, where

$$j = \frac{p-1}{\ell} < N^{1/2}/\log^3 N.$$

For each $j < N^{1/2}/\log^3 N$, we count the number of such p.
This is $O(j)$.

Summing on j gives $O(N/\log^6 N)$ such p.
This is $o(\pi(N)/\phi(W))$.

Maynard–Tao-ification

Fix an admissible set $\{h_1, \ldots, h_k\}$.
We look for primes p among $n + h_1, \ldots, n + h_k$ belonging to $\tilde{\mathcal{P}}$:
primes with 2 as a primitive root.

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We W-trick-it-out:
Let $W = 4 \prod_{p \le D_0} p$.
Choose $\nu \pmod{W}$ so that whenever $n \equiv \nu \pmod{W}$,

- each $n + h_i$ is coprime to W,
- each $n + h_i \equiv 3 \pmod{8}$,
- each $n + h_i - 1$ has no odd prime factors $\le D_0$.

This can be done if 8 divides every h_i.

Maynard's method depends on making S_2/S_1 large, where

$$S_1 = \sum_{\substack{N \le n < 2N \\ n \equiv \nu \pmod{W}}} w(n),$$

$$S_2 = \sum_{\substack{N \le n < 2N \\ n \equiv \nu \pmod{W}}} \Big(\sum_{i=1}^{k} \mathbf{1}_{n+h_i \text{ prime}}\Big) w(n).$$

Let $\tilde{\mathcal{P}}$ be the primes with 2 as a primitive root.

Claim: $\tilde{S}_2 := \sum_{\substack{N \le n < 2N \\ n \equiv \nu \pmod{W}}} \left(\sum_{i=1}^{k} \mathbf{1}_{n+h_i \in \tilde{\mathcal{P}}}\right) w(n)$ obeys the same asymptotic as S_2.

Looking at the difference $S_2 - \tilde{S}_2$, it is enough to make

$$\sum_{\substack{N \le n < 2N \\ n \equiv \nu \pmod{W}}} (\mathbf{1}_{n+h_i \text{ prime}} - \mathbf{1}_{n+h_i \text{ in } \tilde{\mathcal{P}}}) w(n)$$

small, for each fixed $1 \le i \le k$. Fix $i = k$ (notational convenience).

If $p = n + h_k$ is prime but 2 is not a primitive root, then p has P_ℓ for some ℓ.

By our W-tricking, we know $\ell > D_0$.

Split into 4 ranges for ℓ:

I. $D_0 < \ell \leq (\log N)^{100k}$,
II. $(\log N)^{100k} < \ell \leq N^{1/2}(\log N)^{-100k}$,
III. $N^{1/2}(\log N)^{-100k} < \ell \leq N^{1/2}(\log N)^{100k}$,
IV. $N^{1/2}(\log N)^{100k} < \ell$.

We estimate the contribution to $\sum_n (\mathbf{1}_{n+h_k \text{ prime}} - \mathbf{1}_{n+h_k \text{ in } \tilde{\mathcal{P}}})w(n)$ from n with $p = n + h_k$ satisfying P_ℓ for an ℓ in each of these ranges.

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Ranges II and and IV we treat by Cauchy–Schwarz, using that there are not too many $p \leq 3N$ having P_ℓ for some ℓ in that range.

Example

For each ℓ, we get in II an upper bound $\ll \frac{N/\log N}{\ell(\ell-1)} + N^{1/2}\log N$, and summing on ℓ gives

$$\ll N(\log N)^{-100k}.$$

In other words,

$$\sum_n \mathbf{1}_{p=n+h_k \text{ one of these primes}} \ll N(\log N)^{-100k}.$$

We now use the easy bound $\sum_n w(n)^2 \ll N(\log N)^{20k}$.

We get

$$\sum_n \mathbf{1}_{p=n+h_k \text{ one of these primes}} w(n)$$

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Range IV is similarly easy.

Range I: $D_0 < \ell \le (\log N)^{100k}$

To estimate

$$\sum_n \mathbf{1}_{p=n+h_k \text{ one of these primes}} w(n),$$

open up the sum. Have to estimate

$$\sum_{\ell} \sum_{\substack{\mathbf{d},\mathbf{e} \\ d_k=e_k=1}} \lambda_{\mathbf{d}}\lambda_{\mathbf{e}} \sum_{\substack{N\le n<2N \\ n\equiv\nu \pmod W \\ [d_i,e_i]\mid n+h_i\ \forall i}} \mathbf{1}_{p=n+h_k \text{ is prime, has } P_\ell}.$$

Inner sum has main term $\approx \frac{N/\log N}{\ell(\ell-1)\phi(W)\prod_{i=1}^{k}[d_i,e_i]}$; error is under control because outer sum on ℓ is small.

Range I: $D_0 < \ell \leq (\log N)^{100k}$

Inner sum has main term $\approx \frac{N/\log N}{\ell(\ell-1)\phi(W)\prod_{i=1}^{k}[d_i,e_i]}$; error is under control because outer sum on ℓ is small.

Summing the main term on ℓ works out similarly to S_2, except we gain a factor of

$$\sum_{\ell} \frac{1}{\ell(\ell-1)}$$

over $D_0 < \ell \leq (\log N)^{100k}$, and this is $o(1)$.

So this is negligible compared to S_1 and S_2.

Range III: $N^{1/2}(\log N)^{-100k} < \ell \leq N^{1/2}(\log N)^{100k}$

To estimate

$$\sum \mathbf{1}_{p=n+h_k \text{ one of these primes}} w(n),$$

replace

$$\mathbf{1}_{p=n+h_k \text{ one of these primes}}$$

with

$$\mathbf{1}_{n+h_k \equiv 1 \pmod{\ell}}.$$

Opening it up gives a sum similar to S_1, but we gain a factor of

$$\sum_{N^{1/2}(\log N)^{-100k} < \ell \leq N^{1/2}(\log N)^{100k}} \frac{1}{\ell} = o(1).$$

Further examples of Maynard-Tao–ification

Theorem (Thompson and P.)

For each function f among $d(n), \phi(n), \sigma(n), \omega(n), \Omega(n)$, one can find arbitrarily long runs of consecutive primes p on which $f(p-1)$ is increasing. Same for decreasing.

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Theorem

There are arbitrarily long runs of primes p for which $p-1$ is squarefree.

Theorem (Baker and P.)

Assume GRH. Fix an elliptic curve $E/\mathbb{Q}$. There are arbitrarily long runs of primes p for which $E(\mathbb{F}_p)$ is cyclic.

Thank you very much!