Big doings with small g a p s

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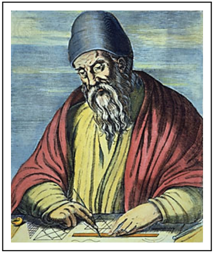
SMALL GAPS: A SHORT SURVEY

300 BCE

Let p_n denote the nth prime number, so $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$.

Theorem (Euclid)

There are infinitely many primes. In other words, if $\pi(x) := \#\{p \le x : p \text{ prime}\}$*, then* $\pi(x) \to \infty$ *as* $x \to \infty$*.*



Proof.

Suppose otherwise. Suppose $p_1, \ldots, p_n$ is the complete list, and let $P := p_1 \cdots p_n + 1$. P leaves a remainder of 1 when divided by each p_i, and so has no prime divisor. But **every** integer > 1 has a prime divisor.

Twin primes

Once we know that there are infinitely many primes, one wants to know where they fall on the number line. One way to get a handle on this is to look at the gap sequence.

Call the nth prime gap d_n, so that

$$d_n = p_{n+1} - p_n.$$

The sequence $\{d_n\}$ begins

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 8, \ldots$$

(`OEIS A001223`)

It was noticed many moons ago that $d_n = 2$ appears to appear infinitely often.

Recall: A pair of prime numbers $\{p, p+2\}$ is called a *twin prime pair*.

Twin prime pairs: $\{3, 5\}$, $\{5, 7\}$, $\{11, 13\}$, $\{17, 19\}$, $\{29, 31\}$, $\{41, 43\}$, $\{59, 61\}$, $\{71, 73\}$, $\ldots$

Twin prime conjecture: There are infinitely many twin prime pairs.

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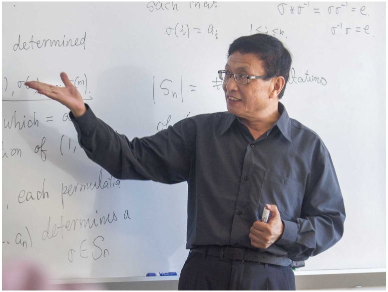
Twin prime conjecture: There are infinitely many twin prime pairs.

No one knows precisely when this very natural conjecture was made. The term *twin prime* itself is relatively recent, having been introduced by Stäckel in 1916 (in German: “Primzahlzwilling”). But the conjecture itself is surely centuries older.

A desperate professor, a brilliant student and a 2000-year-old math problem collide in this thriller about ambition, ego and the nature of genius.

Combining work of Goldston–Pintz–Yildirim (2005) with groundbreaking new results on how primes are distributed in arithmetic progressions, Zhang proved a startlingly strong approximation to the twin prime conjecture.



Theorem (Y. Zhang, April 2013)

There are infinitely many values of n for which $d_n < 70 \cdot 10^6$.

Theorem (J. Maynard, November 2013)

$d_n \leq 600$ infinitely often.

The method used by Maynard was discovered almost simultaneously by Tao. Their results were refined in work of the `Polymath8b` team, who showed that

$$d_n \leq 246$$

infinitely often.

`Polymath8b` also improved the known results on gaps between primes two apart in the sequence of primes, instead of consecutive. For example, they showed that infinitely often this gap is at most

$$398130.$$

(MORE) BIG DOINGS

Primes and periodicity

Consider the following chart of the numbers 1–26 arranged into 2 rows:

1	3	5	7	9	11	13	15	17	19	21	23	25
2	4	6	8	10	12	14	16	18	20	22	24	26

Let's color in the primes.

Primes and periodicity

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2	4	6	8	10	12	14	16	18	20	22	24	26

The second row has only the prime 2; every other prime appears in the first row.

There's no mystery about why this is the case: Prime numbers are odd!

What if we try more than two rows?

With 6 rows, the chart now looks like ...

1	7	13	19	25	31	37	43	49	55	61	67	73
2	8	14	20	26	32	38	44	50	56	62	68	74
3	9	15	21	27	33	39	45	51	57	63	69	75
4	10	16	22	28	34	40	46	52	58	64	70	76
5	11	17	23	29	35	41	47	53	59	65	71	77
6	12	18	24	30	36	42	48	54	60	66	72	78

Again, let’s color in the primes.

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5	11	17	23	29	35	41	47	53	59	65	71	77
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In this case, there are only two rows with more than one prime. In the second, fourth, and sixth row, the numbers all have a factor of 2. And in the third row, they all have a factor of 3.

In general, if we arrange the natural numbers into n rows, if the row number has a common factor with n, so will every number in the row. Those rows can contain at most one prime.

For the other rows, there's no reason they couldn't contain many primes. Call such rows *admissible*.

Theorem (Dirichlet, 1837)

Every admissible row contains infinitely many primes.

Dirichlet's theorem answers a very natural question about how the primes fall into these rows. But it's not the end of the story.

Example

Let's go back to our arrangement into six rows, where the admissible rows were the first and fifth.

The pair $5, 7$ is a pair of *consecutive* primes where the first is from row 5 and the second is from row 1.

This happens again: $11, 13$. And again: $17, 19$.

Does this happen infinitely often?

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Does this happen infinitely often?

Yes, we have to switch back and forth like this infinitely often, since both rows contain infinitely many primes.

Example

This easy question suggests a harder question.

The primes 23 and 29 are a pair of consecutive primes both from the fifth row. This happens again: 53 and 59.

Does it happen infinitely often?

It’s not so obvious anymore.

Shiu strings 

Theorem (D. K. L. Shiu, 2000)

Arrange the positive integers into n rows. Pick any admissible row. There are infinitely many pairs of consecutive primes which land in that row.

In fact, this is true not just for pairs of consecutives, but for triples, quadruples, or tuples of any length.

Example

You can find 10^6 consecutive primes that all end in the digit 7.

What is Shiu's theorem doing in this talk?

The new work on bounded gaps gives another proof of Shiu's beautiful result. The proof is much simpler than the original argument.

Moreover, the new methods give more than what Shiu was able to show.

Example

We mentioned that there are infinitely many runs of consecutive primes of length 10^6 that end in 7. These new methods imply that a positive proportion of all primes start such a run (Maynard, 2014).

Another question of Gauss

If p is a prime other than 2 or 5, the decimal expansion of $\frac{1}{p}$ is purely periodic. For example,

$$\frac{1}{3} = 0.333333\cdots = 0.\overline{3},$$

$$\frac{1}{7} = 0.\overline{142857}, \quad \frac{1}{11} = 0.\overline{09}, \quad \text{and} \quad \frac{1}{13} = 0.\overline{076923}.$$

Questions

What is the length of the period here?

He knew it was always a divisor of $p - 1$. Sometimes it's equal, as when $p = 7$, and sometimes not (as in the other examples above).

Questions (Gauss, ca. 1800)

Is the period of $\frac{1}{p}$ equal to $p-1$ infinitely often?

Such primes have an interesting property: If you know the expansion of $\frac{1}{p}$, then you get the expansions of $\frac{2}{p}, \frac{3}{p}, \ldots, \frac{p-1}{p}$ just by shifting; e.g.,

$$\frac{1}{7} = 0.\overline{142857}$$

while

$$\frac{5}{7} = 0.\overline{571428}.$$

Gauss did extensive computations suggesting an affirmative answer.

In fact, it seems that about 37% of the primes have Gauss's property. Here's a 51-digit example:

100000000000000000000000000000000000000000000000709.

Seek and ye shall find!

Nevertheless, Gauss's guess is still unproved.

The problem is connected with the **Generalized Riemann Hypothesis**, an unproved statement that is nevertheless often assumed in number theoretic investigations.

Assuming the Generalized Riemann Hypothesis, Hooley showed in 1967 that Gauss was right.

Using the machinery introduced by Maynard and Tao, and making the same hypothesis, you can find infinitely many pairs, triples, quadruples, etc., of consecutive primes all of which have Gauss's property. (Pollack 2014)

Some questions of Sierpiński

Let $s(n)$ denote the sum of the decimal digits of n. For example, $s(2014) = 2 + 1 + 4 = 7$. We can observe that

$$s(1442173) = s(1442191) = s(1442209) = s(1442227).$$

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Questions (Sierpiński, 1961)

Given m, are there infinitely many m-tuples of consecutive primes $p_n, \ldots, p_{n+m-1}$ with

$$s(p_n) = s(p_{n+1}) = \cdots = s(p_{n+m-1})?$$

Answer (Thompson  **and P.)**: Yes.

Other number systems

The Maynard–Tao work on gaps between primes can be ported over to other settings, as long as those settings share enough properties in common with the usual setting of positive integers.

Example

Define a *B***-number** as a finite nonempty sequence of 0s and 1s, with no leading 0s unless the string consists only of 0. Listing strings by length, the first few examples are thus 0, 1, 10, 11, 101, $\ldots$. We define **non-carry addition** $(+)$ and **non-carry multiplication** $(\times)$ of *B*-numbers by the usual grade-school algorithms for addition and multiplication but systematically ignoring carries. For example, $1 + 1 = 0$ with our definition.

And...

$$\begin{array}{rr} & 10101 \\ + & 1101 \\ \hline & 11000 \end{array} \qquad \text{while} \qquad \begin{array}{rr} & 10101 \\ \times & 1101 \\ \hline & 10101 \\ & 00000 \\ & 10101 \\ & 10101 \\ \hline & 11101001 \end{array}$$

A **prime** *B*-number is one with more than one digit which cannot be written as a non-carry product except as $1 \times$ itself or itself $\times\ 1$. For example, 10 and 11 are prime, but 11101001 is not.

Questions

Is there a bounded gaps theorem for prime B-numbers?

Yes!

Theorem

There are infinitely many pairs of prime B-numbers which differ only in their last 9 digits.

This is worked out by Castillo, Hall, Lemke Oliver, Thompson and P. (2014).

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Where do B-numbers come from? The multiplication on B-numbers is the same as the multiplication on polynomials

$$a_d x^d + \cdots + a_2 x^2 + a_1 x + a_0$$

but with coefficients always reduced modulo 2. Similar theorems hold if the coefficients are reduced modulo p for any fixed prime p.

Much more could be said, and many more people's work could be mentioned (Banks, Freiberg, Turnage-Butterbaugh, Thorner, Baker, ...), but now seems like a suitable place to stop.

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THANK YOU!