ON THE GREATEST COMMON DIVISOR OF A NUMBER AND ITS SUM OF DIVISORS, II

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In memory of Eduard Wirsing, with appreciation and admiration.

Abstract. Let $E(x,y) = \#\{n \le x : \gcd(n, \sigma(n)) > y\}$. We collect known results about the distribution of $E(x,y)$ and establish a new, sharp estimate for $E(x,y)$ when y grows faster than any power of $\log\log x$ but $y = \exp((\log\log x)^{o(1)})$. Taken together, these results determine the order of magnitude of $\log(E(x,y)/x)$ whenever $1 \le y \le x^{1-\epsilon}$.

1. Introduction

1.1. **Perfect numbers.** A natural number n is called **perfect** if $\sigma(n) = 2n$; equivalently, n is perfect if n is the sum of its proper divisors. Perfect numbers appear already in Euclid's *Elements* (ca. 300 BCE), where it is shown that $2^{k-1}(2^k - 1)$ is perfect whenever $2^k - 1$ is prime. Two thousand years later, Euler established a partial converse to Euclid's theorem: *Every* even *perfect number is given by Euclid's formula.*

To this day, no odd perfect numbers are known, and deciding whether any exist stands as perhaps the oldest unsolved problem in number theory. More modestly, one might hope to show that if odd perfect numbers exist, at least there cannot be too many of them. To quantify this, let $V(x)$ denote the number of perfect numbers $n \le x$. (While even perfect numbers are counted in $V(x)$, the count of even perfect numbers in $[1,x]$ is $O(\log x)$, which is dwarfed by all of the upper bounds on $V(x)$ to be discussed.) In 1933, Davenport [2] showed that $n/\sigma(n)$ has a continuous distribution function. That is, for each $u \in [0,1]$, the asymptotic density of n with $n/\sigma(n) \le u$ exists, and this density varies continuously with u. The continuity in Davenport's result implies that for any fixed real number μ, the n with $n/\sigma(n) = \mu$ make up a set of density 0. In particular ($\mu = \frac{1}{2}$), $V(x) = o(x)$, as $x \to \infty$. In 1954, Kanold gave a more direct proof that $V(x) = o(x)$ [10]. Kanold's contemporaries seem to have viewed his paper as throwing down the proverbial gauntlet, prompting a flurry of improved bounds for $V(x)$ over the next several years, collected in Proposition 1.

Proposition 1. *We have the following upper bounds for $V(x)$. All O-estimates are to be understood as holding when x is sufficiently large.*

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Researcher(s)	**Year**	**Estimate**
Volkmann [20]	*1955*	$V(x) = O(x^{5/6})$
Hornfeck [8]	*1955*	$V(x) < x^{1/2}$ *for all* $x > 0$
Kanold [11]	*1956*	$V(x) = o(x^{1/2})$, *as* $x \to \infty$
Erdős [6]	*1956*	$V(x) = O(x^{1/2-\delta})$, *for some* $\delta > 0$
Kanold [12]	*1957*	$V(x) = O(x^{1/4} \log x / \log\log x)$
Hornfeck and Wirsing [9]	*1957*	$V(x) = O_\epsilon(x^\epsilon)$; *in fact,* $V(x) \le x^{O(\frac{\log\log\log x}{\log\log x})}$
Wirsing [21]	*1959*	$V(x) \le x^{O(1/\log\log x)}$

Incredibly, Wirsing is still 'winner and world champion' as far as upper bounds on $V(x)$: Despite six decades of further investigations, we still have no proof that $V(x) \le x^{\epsilon(x)}$ for a function $\epsilon(x) = o(1/\log\log x)$.

It seems appropriate given the nature of this volume to sketch a version of Wirsing's ingenious argument. Suppose that $n \le x$ is perfect, and suppose also that we have in hand a unitary divisor[1] d of n with $d > 1$. Then either $\sigma(d) = 2d$, in which case $n = d$, or $d < \sigma(d) < 2d$, in which case $\frac{2d}{\sigma(d)}$ has a lowest-terms denominator larger than 1. Since $\frac{2d}{\sigma(d)} = \frac{\sigma(n/d)}{n/d}$, if we choose p_1 as the least prime dividing the denominator of $\frac{2d}{\sigma(d)}$ (which, it should be noted, depends only on d), then p_1 divides n/d. We let e_1 be the positive integer for which which $p_1^{e_1} \parallel n/d$ and start the argument over with our unitary divisor d replaced by the new unitary divisor $dp_1^{e_1}$. We continue in the same way until our unitary divisor reaches n itself, at which point we have discovered a factorization

$$n = dp_1^{e_1} \cdots p_k^{e_k}$$

where each p_i is entirely determined by of d and the exponents e_j for $j < i$. Then n itself is determined by d and the exponent sequence $e_1, \dots e_k$. Wirsing takes d as the $\log x$-smooth part of n (which, as is not difficult to show, must exceed 1 once x is large), and he derives his $x^{O(1/\log\log x)}$ estimate by bounding above the number of choices for d and for the exponent sequence $e_1, \dots, e_k$.

Wirsing's results in [21] extend somewhat beyond an upper bound on $V(x)$. What is actually shown is that *every* equation $\sigma(n) = \lambda n$ has at most $x^{O(1/\log\log x)}$ solutions $n \le x$, once $x \ge 3$, where the implied constant is independent of λ. The independence of the bound on λ can be crucial in applications (such as Lemma 8 below; another example is the main result of the recent paper [15]). One easy consequence of this uniformity is that the number of $n \le x$ that are **multiply perfect** — meaning that $\sigma(n)/n \in \mathbb{Z}$ — is also bounded by $x^{O(1/\log\log x)}$.

1.2. **The distribution of** $\gcd(n, \sigma(n))$**.** The just mentioned consequence of Wirsing's theorem for multiply perfect numbers can be read as saying that there are very few n with $\gcd(n, \sigma(n))$ as large as possible. In this note we are interested more generally in the distribution of $\gcd(n, \sigma(n))$ as n ranges through the integers in $[1, x]$.

[1] meaning, $d \mid n$ and $\gcd(d, n/d) = 1$

It was Erdős who opened up this line of investigation. In [5], Erdős shows that the number of $n \le x$ with $\gcd(n, \sigma(n)) = 1$ is $\sim \mathrm{e}^{-\gamma} x / \log\log\log x$, as $x \to \infty$, where γ is the Euler–Mascheroni constant.[2] In §2, we present a souped-up version of Erdős's argument, proving that the count of $n \le x$ with $\gcd(n, \sigma(n)) = m$ is $\sim \mathrm{e}^{-\gamma} x / m \log\log\log x$, uniformly for $m \le (\log\log x)^{1/4}$.

Larger values of $\gcd(n, \sigma(n))$, but still of size bounded by a power of $\log\log x$, were considered by Erdős in [6]. To ease notation, define

$$E(x, y) = \#\{n \le x : \gcd(n, \sigma(n)) > y\}.$$

Theorem 4 of [6] asserts the existence of a continuous function $D(u)$, strictly decreasing on $(0, \infty)$, such that for each positive real number u,

$$E(x, (\log\log x)^u) \sim D(u)x, \tag{1}$$

as $x \to \infty$. While the proof is omitted in [6], details appear in later joint work with Luca and Pomerance [7] where it is shown that

$$D(u) = \mathrm{e}^{-\gamma} \int_u^\infty \rho(t)\, \mathrm{d}t, \tag{2}$$

with $\rho(u)$ the **Dickman function** from 'smooth number' theory: the solution on $(0, \infty)$ to the difference delay equation $\rho'(u) = -\rho(u-1)/u$ for $u > 1$, with $\rho(u) = 1$ for $0 < u \le 1$. The relation (1) is a consequence of another result of [7], of independent interest, that $\gcd(n, \sigma(n))$ is the $\log\log x$-smooth part of n for all but $o(x)$ values of $n \le x$.[3] Though not stated explicitly there, the argument in [7] establishes that the asymptotic relation (1) holds uniformly in u, for u restricted to any compact subinterval of $(0, \infty)$.

Another claim of [6], again stated without proof (see Theorem 3 there and the subsequent remarks), is that $E(x, y)$ undergoes a phase transition as y grows beyond $y = (\log x)^{o(1)}$. A corrected version of this claim is established in [14], where it is shown that the true threshold is $y = \exp((\log\log x)^{o(1)})$.

Proposition 2 (see Theorems 1.1 and 1.2 in [14])**.** *If $x \to \infty$ and $y = \exp((\log\log x)^{o(1)})$, then*

$$E(x, y) > x / y^{o(1)},$$

while for each $\beta > 0$ there is a constant $c = c(\beta) > 0$ with

$$E(x, y) < x/y^c \quad \textit{whenever } y > \exp((\log\log x)^\beta) \textit{ and } x \textit{ is large.}$$

As an illustration of the second half of Proposition 2, if $y > \exp((\log\log x)^{1/3})$, one can deduce from the proofs in [14] that $E(x, y) < xy^{-1/1000}$ once x is sufficiently large. The shape of the upper bound — x divided by a constant power of y — is best possible, up to the precise constant, since it is shown in [14, Theorem 1.4] that $E(x, y) > x/y^{1+o(1)}$ when $x \to \infty$ and $2 \le y \le x^{1-\epsilon}$ (for any fixed $\epsilon > 0$).

[2]Erdős states his result for $\gcd(n, \varphi(n))$ rather than $\gcd(n, \sigma(n))$, but the argument for $\sigma(n)$ is very similar. See also [17].

[3]These results of [7] are stated for $\varphi(n)$, but the proofs for $\sigma(n)$ are essentially the same.

The above results give only weak information about $E(x,y)$ when y tends to infinity faster than any power of $\log\log x$ but $y = \exp((\log\log x)^{o(1)})$. Our main theorem addresses this missing range.

Theorem 3. *If $x \to \infty$ and $u := \frac{\log y}{\log\log\log x} \to \infty$, with $y \le \exp((\log\log x)^{o(1)})$, then*

$$E(x,y) = x\exp\left(-\left(1+o(1)\right)u\log u\right). \tag{3}$$

We also prove a somewhat weaker estimate in the wider range $y \le \exp((\log\log x)^{1-\epsilon})$.

Theorem 4. *Fix $\epsilon > 0$. If $x \to \infty$ and $u := \frac{\log y}{\log\log\log x} \to \infty$, with $y \le \exp((\log\log x)^{1-\epsilon})$, then*

$$x\exp\left(-\left(1+o(1)\right)u\log u\right) \le E(x,y) \le x\exp\left(-\left(1/7+o(1)\right)u\log u\right). \tag{4}$$

We do not know if (3) holds in the entire range of Theorem 4.

The function $D(u)$, as defined in (2), can be shown to satisfy $D(u) = \exp(-(1+o(1))u\log u)$ as $u \to \infty$ (cf. the arguments of §3). Thus Theorems 3 and 4 assert that weaker versions of (1) hold in extended ranges of y. For the lower bounds, and for the upper bound when y is small, we prove Theorems 3 and 4 by borrowing ideas from the proof of (1). For larger values of y, we obtain the upper bounds by adapting the method used by Erdős in [6] to bound the counts of perfect and multiperfect numbers. These arguments of Erdős were also the basis of much of the work in [14].

In [14], Wirsing's theorem is used to deduce that $\frac{1}{x}\sum_{n\le x}\gcd(n,\sigma(n)) \le x^{O(1/\sqrt{\log\log x})}$. (This result is quoted as Lemma 8 below.) Thus if y tends to infinity faster than any power of $x^{1/\sqrt{\log\log x}}$, then $E(x,y) < x/y^{1+o(1)}$. That requirement on y is surely too stringent; it would be very interesting to know the true threshold for y after which the savings of $y^{1+o(1)}$ 'kicks in'. Perhaps it suffices for $\log y$ to grow faster than any power of $\log\log x$. Perhaps even $y > \exp((\log\log x)^{1+\epsilon})$ is enough? It follows from Theorem 4 that $y > \exp((\log\log x)^{1-\epsilon})$ is *not* sufficient.

A word on notation. We write $A \gtrsim B$ to mean $A \ge (1+o(1))B$; naturally, $A \lesssim B$ means $B \gtrsim A$. The letters p and ℓ (but *not* q) are reserved for primes.

2. The frequency of n with $\gcd(n,\sigma(n)) = m$

In this section we prove the claim made in the introduction that

$$\#\{n \le x : \gcd(n,\sigma(n)) = m\} \sim \mathrm{e}^{-\gamma}\frac{x}{m\log\log\log x}, \tag{5}$$

as $x \to \infty$, uniformly for $m \le (\log\log x)^{1/4}$. We have not seen this result in the literature, but the method of proof follows [5] and [7] closely. The argument is included for completeness, and to clear ground for the proofs of Theorems 3 and 4 in §3.

The following lemma is due to Pomerance (see [16, Theorem 3] and its application on p. 221 there).

Lemma 5. *For all $x \ge 3$ and each positive integer $d \le x$, the number of $n \le x$ for which $d \nmid \sigma(n)$ is $O(x/(\log x)^{1/\varphi(d)})$. Here the implied constant is absolute.*

Put

$$Z := \frac{\log\log x}{2\log\log\log x}, \tag{6}$$

and let

$$L = \operatorname{lcm}\{d : 1 \le d \le Z\}. \tag{7}$$

Let m be a divisor of L not exceeding $x^{1/2}$. (This certainly allows all $m \le (\log\log x)^{1/4}$, once x is large. The extra generality will be useful later.)

We consider $n \le x$ of the form $n = mq$, where every prime dividing q exceeds Z. Performing inclusion-exclusion over the primes dividing Z, one finds that there are $\sim \frac{x}{m}\prod_{\ell \le Z}(1 - 1/\ell) \sim$ $\mathrm{e}^{-\gamma} x/m \log\log\log x$ such n. By Lemma 5, the number of these n for which $\sigma(q)$ is *not* divisible by L is

$$\ll \frac{x}{m}\sum_{d \le Z}(\log(x/m))^{-1/\varphi(d)} \ll \frac{x}{m}\sum_{d \le Z}(\log x)^{-1/Z} \le \frac{x}{m\log\log x},$$

which is $o(x/m\log\log\log x)$. Hence, there are $\sim \mathrm{e}^{-\gamma}x/m\log\log\log x$ values $n = mq \le x$ with q having all prime divisors greater than Z and with $\sigma(q)$ divisible by L. All of these n are such that $m \mid \gcd(n, \sigma(n))$.

From now on, we assume that $m \le (\log\log x)^{1/4}$.

Let us show that almost all of the n constructed above have $\gcd(n, \sigma(n)) = m$. If m is a proper divisor of $\gcd(n, \sigma(n))$, then $\ell \mid \gcd(n, \sigma(n))$ for some prime $\ell > Z$. Since $Z > \sigma(m)$, it must be that this prime ℓ divides $\gcd(q, \sigma(q))$. We may assume q is squarefree; otherwise n has squarefull part at least Z^2, and the number of such n is $O(x/Z)$, which is $o(x/m\log\log\log x)$. Thus there are primes $\ell, p > Z$ dividing q with $p \equiv -1 \pmod{\ell}$. The number of such $q \le x/m$ for which $\ell > Z' := \log\log x \cdot \log\log\log x$ is

$$\ll \frac{x}{m}\sum_{\ell > Z'}\sum_{\substack{p \le x \\ p \equiv -1 \ (\mathrm{mod}\ \ell)}}\frac{1}{p\ell} \ll \frac{x}{m}\sum_{\ell > Z'}\frac{\log\log x}{\ell^2} \ll \frac{x}{m(\log\log\log x)^2}, \tag{8}$$

which is also $o(x/m\log\log\log x)$. (We used here that $\sum_{p \le x,\ p \equiv -1\ (\mathrm{mod}\ \ell)} 1/p \ll (\log\log x)/\ell$, which follows from the Brun–Titchmarsh theorem by partial summation.) If instead $\ell \in (Z, Z']$, write $q = \ell q'$. Since $q' \le x/m\ell$ and q' is free of prime factors up to Z, there are $\ll x/m\ell\log\log\log x$ possibilities for q' given ℓ. Summing on $\ell \in (Z, Z']$ shows that there are $o(x/m\log\log\log x)$ integers $n = mq$ of this kind as well. Collecting our results we arrive at the lower bound implicit in (5).

To prove the upper bound, suppose $\gcd(n, \sigma(n)) = m$, and write $n = mq$. It suffices to show that, with $o(x/m\log\log\log x)$ exceptions, q is free of prime factors below Z. Write $q = q_1q_2$, where every prime dividing q_1 divides m and $\gcd(q_2, m) = 1$. If $q_1 > m^2(\log\log\log x)^3$, then n has a squarefull divisor larger than $m^2(\log\log\log x)^3$, putting n in a set of size

$o(x/m \log\log\log x)$. So we may suppose that $q_1 \le m^2(\log\log\log x)^3$. We may also suppose that $L \mid \sigma(q_2)$, since the number of exceptional q is

$$\ll \sum_{q_1}\sum_{d\le Z} \frac{x/mq_1}{(\log(x/mq_1))^{1/\varphi(d)}} \ll \frac{x}{m}\sum_{q_1}\frac{1}{q_1}\sum_{d\le Z}(\log x)^{-1/Z} \ll \frac{x}{m\log\log x}.$$

Since $mq_1 < Z$ and $L \mid \sigma(q_2) \mid \sigma(n)$, we see that $mq_1 \mid \gcd(n,\sigma(n)) = m$, forcing $q_1 = 1$. Thus, $q = q_2$; that is, q is prime to m. If q has a prime factor $p \le Z$, then $p \nmid m$, and $pm \mid L$ (since both $p, m \mid L$). But then $pm \mid \gcd(n,\sigma(n))$, contradicting that $\gcd(n,\sigma(n)) = m$.

3. Proof of Theorem 3

3.1. **Preliminaries concerning smooth numbers.** We begin by collecting certain statements from 'smooth number' theory that will prove useful momentarily.

Let $\Psi(X,Y)$ denote the count of Y-smooth $n \le X$, and let $\Psi_2(X,Y)$ denote the same count restricted to squarefree n. Fix $\epsilon > 0$. It is known that whenever $X, Y \to \infty$ with $U := \frac{\log X}{\log Y} \to \infty$ and $Y \ge (\log X)^{1+\epsilon}$, we have

$$X\exp(-(1+o(1))U\log U) \le \Psi_2(X,Y) \le \Psi(X,Y) \le X\exp(-(1+o(1))U\log U),$$

where $U := \log X/\log Y$. Here the upper bound follows from [3, Theorem 2] while the lower bound is a consequence of the lower bound on $\Psi(X,Y)$ in [1, Theorem 3.1] combined with [13, Théorème 2], which estimates the ratio $\Psi(X,Y)/\Psi_2(X,Y)$. See also [4, Théorème 2.1].

Recall also that

$$\Psi(X,Y) \sim X\rho(U) \quad \text{if } X, Y \to \infty \text{ with } X \ge Y \ge \exp((\log\log X)^2); \tag{9}$$

furthermore,

$$\rho(U) = \exp(-(1+o(1))U\log U), \quad \text{as } U \to \infty, \quad \text{and} \tag{10}$$

$$\rho(U+1) \sim \frac{\rho(U)}{U\log U}, \quad \text{as } U \to \infty. \tag{11}$$

For proofs of (9)–(11), see Chapter III.5 of [19]; the relation (11) is proved by combining equations (5.48) and (5.62) there.

3.2. **Lower bounds in Theorems 3 and 4.** We shall prove that $E(x,y) \ge x\exp(-(1+o(1))u\log u)$ whenever $u = \log y/\log\log\log x \to \infty$ and $y \le \exp((\log\log x)^{1-\epsilon})$.

To start off, assume that $u \ge \log\log\log\log x$; that is, $y \ge (\log\log x)^{\log\log\log\log x}$.

Let Z and L be as defined in (6) and (7). Consider n of the form $n = mq$, where $m \in (y, x^{1/2}]$ is both squarefree and Z-smooth, and where $q \le x/m$ has all prime factors exceeding Z. By our work in §3, for each m there are $\sim e^{-\gamma}x/m\log\log\log x$ corresponding values of q, and this remains true if we also require that L divides $\sigma(q)$. Then $m \mid L \mid \sigma(q) \mid \sigma(n)$, so that $\gcd(n,\sigma(n)) \ge m > y$. Moreover, the number of n produced this way is

$$\sim e^{-\gamma}\frac{x}{\log\log\log x}\sum_{\substack{m\in(y,x^{1/2}]\\ m\ Z\text{-smooth}}}\frac{\mu^2(m)}{m}.$$

Since $\exp(u\log u) \ge (\log\log\log x)^{\log\log\log\log\log x}$, the denominator $\log\log\log x$ has the shape $\exp(o(u\log u))$. Thus, it suffices to bound the sum from below by $\exp(-(1+o(1))u\log u)$.

Put $U = \log y/\log Z$, so that $U \lesssim (\log\log x)^{1-\epsilon}/\log\log\log x$. We fix $\delta > 0$ and define $y' = y\exp(\delta U \log U)$. Then $y' \le \exp((\log\log x)^{1-\epsilon+o(1)})$. Letting $U' = \log y'/\log Z$, we see that

$$U' - U = \frac{\delta U \log U}{\log Z} \lesssim \delta(1-\epsilon)U.$$

Hence, $U' \lesssim (1+\delta(1-\epsilon))U$ and $U'\log U' \lesssim (1+\delta(1-\epsilon))U\log U$. Thus,

$$\frac{\Psi_2(y',Z)}{\Psi_2(y,Z)} \ge \frac{\Psi_2(y',Z)}{\Psi(y,Z)} \ge \frac{y'}{y}\exp(U\log U - U'\log U' + o(U\log U)),$$

which is at least $\exp((\delta\epsilon + o(1))U\log U)$, and hence (eventually) larger than 2. Therefore,

$$\sum_{\substack{m\in(y,x^{1/2}]\\ m\ Z\text{-smooth}}} \frac{\mu^2(m)}{m} \ge \sum_{\substack{m\in(y,y']\\ m\ Z\text{-smooth}}} \frac{\mu^2(m)}{m} \ge \frac{1}{y'}(\Psi_2(y',Z) - \Psi_2(y,Z))$$

$$\ge \frac{1}{2y'}\Psi_2(y',Z) \ge \exp(-(1+o(1))U'\log U').$$

Since $U' \lesssim (1+\delta(1-\epsilon))U$ and $U \sim u$, we conclude that $\sum_{m\in(y,x^{1/2}],\ m\ Z\text{-smooth}} \mu^2(m)/m$ is eventually larger than $\exp(-(1+\delta)u\log u)$. But $\delta > 0$ was arbitrary, and so

$$\sum_{\substack{m\in(y,x^{1/2}]\\ m\ Z\text{-smooth}}} \frac{\mu^2(m)}{m} \ge \exp(-(1+o(1))u\log u),$$

as desired.

A more careful argument is required when $y \le (\log\log x)^{\log\log\log\log x}$.

We consider $n \le x$ of the form $n = mq$, where m is a divisor of L from the interval $(y, x^{1/2}]$, q has all prime factors exceeding Z, and $L \mid \sigma(q)$. Each such n has $\gcd(n,\sigma(n)) \ge m > y$, and our earlier arguments show that the number of such $n \le x$ is

$$\sim \mathrm{e}^{-\gamma}\frac{x}{\log\log\log x}\sum_{\substack{m\mid L\\ y<m\le x^{1/2}}}\frac{1}{m}, \tag{12}$$

as $x \to \infty$.

The sum on m in (12) is majorized by the sum over all Z-smooth $m \in (y, x^{1/2}]$. In fact, these two sums are quite close. To see this, observe that if m is Z-smooth but $m \nmid L$, then there is a prime power $p^e \parallel m$ with $p \le Z$ but $p^e > Z$ (forcing $e > 1$). Thus $m = m_0 m_1$, where m_0 is squarefull and larger than Z, and m_1 is Z-smooth. But the reciprocal sum of all such m is at most

$$\sum_{\substack{m_0>Z\\ \text{squarefull}}}\frac{1}{m_0}\sum_{\substack{m_1\\ Z\text{-smooth}}}\frac{1}{m_1} = \left(\prod_{\ell\le Z}(1-1/\ell)^{-1}\right)\sum_{\substack{m_0>Z\\ \text{squarefull}}}\frac{1}{m_0} \ll \frac{\log Z}{\sqrt{Z}} \ll \frac{1}{(\log\log x)^{1/3}}.$$

Turning to the analogous sum over Z-smooth m, we have that

$$\sum_{\substack{m\ Z\text{-smooth}\\ y<m\le x^{1/2}}} \frac{1}{m} \ge \sum_{\substack{m\ Z\text{-smooth}\\ y<m\le yZ}} \frac{1}{m} \ge -\frac{\Psi(y,Z)}{y} + \int_y^{yZ} \frac{\Psi(v,Z)}{v^2}\,\mathrm{d}v.$$

Let $U = \log y/\log Z$. Then (writing $v = Z^w$ and recalling (9))

$$\int_y^{yZ} \frac{\Psi(v,Z)}{v^2}\,\mathrm{d}v \sim \log Z \int_U^{U+1} \rho(w)\,\mathrm{d}w;$$

clearly this integral over w has size at least $\rho(U+1)$, which is $\sim \rho(U)/U\log U$. Hence, the initial integral over v has size $\gtrsim \rho(U)\log Z/U\log U$. On the other hand, $\Psi(y,Z)/y \lesssim \rho(U) = o(\rho(U)\log Z/U\log U)$. Therefore,

$$\sum_{\substack{m\ Z\text{-smooth}\\ y<m\le x^{1/2}}} \frac{1}{m} \gtrsim \frac{\rho(U)}{U\log U}\log Z \gtrsim \frac{\rho(U)}{U\log U}\log\log\log x. \tag{13}$$

Since $U \lesssim \log\log\log\log x$, we find that $(\log\log x)^{-1/3} = o\left(\frac{\rho(U)}{U\log U}\log\log\log x\right)$. We may thus conclude that the right side of (13) also serves as a lower bound on $\sum_{m\mid L,\ y<m\le x^{1/2}} 1/m$. Plugging this estimate into (12), we complete the proof by noting that $\rho(U) = \exp(-(1+o(1))U\log U) = \exp(-(1+o(1))u\log u)$ while $\mathrm{e}^{\gamma}/(U\log U) = \exp(o(u\log u))$.

3.3. **Upper bound in Theorem 4.** We start by establishing a squarefree variant of the upper bound in Theorem 4. Let

$$E^*(x,y) = \#\{\text{squarefree } n \le x : \gcd(n,\sigma(n)) > y\}.$$

We will show, under the same hypotheses on x, y as Theorem 4, that

$$E^*(x,y) \le x\exp\left(-\left(\frac{1}{3}+o(1)\right)u\log u\right). \tag{14}$$

The claimed upper bound for $E(x,y)$ will be deduced from (14) at the end of this section.

Suppose first that $u \le (\log\log\log x)^{1/2}$. Let $n \le x$ be squarefree with $\gcd(n,\sigma(n)) > y$. We can assume that $\gcd(n,\sigma(n))$ has all of its prime factors at most

$$Z'' := (\log\log x)\exp((\log\log\log x)^{2/3}).$$

Indeed, if $\ell \mid \gcd(n,\sigma(n))$ with $\ell > Z''$, there must be a prime $p \mid n$ for which $p \equiv -1 \pmod{\ell}$. The number of n of this kind is (cf. (8)) bounded by

$$x\sum_{\ell>Z''}\frac{1}{\ell}\sum_{\substack{p\equiv -1 \ (\mathrm{mod}\ \ell)\\ p\le x}}\frac{1}{p} \ll x\sum_{\ell>Z''}\frac{\log\log x}{\ell^2} \ll \frac{x}{\exp((\log\log\log x)^{2/3})},$$

which is dominated by $x\exp(-u\log u)$ in this range of u. But if $\gcd(n,\sigma(n))$ is Z''-smooth, then n has a Z''-smooth divisor exceeding y. In this situation we can invoke the following estimate of Tenenbaum, which is a special case of Exercise 293 on pp. 554–555 of [19]. (Alternatively, we could apply [18, Lemme 3].)

Proposition 6. *Let $X \geq Y' \geq Y \geq 2$. The number of $n \leq X$ whose Y-smooth part exceeds Y' is*

$$\ll x\exp(-V\log V) + x/Y'^{1/2},$$

where $V := \log Y'/\log Y$.

Taking $X = x$, $Y = Z''$, and $Y' = y$ in Proposition 6 bounds the number of n as above as $\ll x\exp(-(1+o(1))u\log u) + x/y^{1/2}$. Since $y = (\log\log x)^u > u^{2u}$, this is at most $x\exp(-(1+o(1))u\log u)$. So we have the upper bound (14) in this case, with the better constant 1 replacing $\frac{1}{3}$.

Suppose instead that $u > (\log\log\log x)^{1/2}$. For this case we need the following lemmas, proved in [14].

Lemma 7 (see [14, Lemma 2.4])**.** *There is an absolute constant C such that the following holds. For each $x \geq 3$ and each squarefree number d, the number of squarefree $n \leq x$ for which $d \mid \sigma(n)$ is at most*

$$\frac{x}{\varphi(d)}(C\omega(d)\log\log x)^{\omega(d)}.$$

Lemma 8 (see [14, Theorem 1.3])**.** *For all $T \geq 3$, we have*

$$\sum_{n\leq T}\gcd(n,\sigma(n)) \leq T^{1+O(1/\sqrt{\log\log T})}.$$

Remark. We do not require the full force of Lemma 8; we could get by with an upper bound of $T^{1+o(1)}$ for the same sum restricted to squarefree n. Such an estimate can be shown in a simpler way (cf. the proof of the upper bound in Theorem 11 of [7]).

Let $d = \gcd(n,\sigma(n))$, and observe that if $n = de$, then $d \mid \sigma(d)\sigma(e)$, and so $\frac{d}{\gcd(d,\sigma(d))} \mid \sigma(e)$.

We take cases according to the size of $w := \omega(d/\gcd(d,\sigma(d)))$. Suppose first that $w \leq \frac{1}{3}u$. Since $n = de$ with $e \leq x/d$ and $\frac{d}{\gcd(d,\sigma(d))} \mid \sigma(e)$, the number of possibilities for n given d is bounded by

$$\frac{x}{d\cdot\varphi(d/\gcd(d,\sigma(d)))}\left(C\cdot\frac{1}{3}u\log\log x\right)^{\frac{1}{3}u}.$$

Since $u \leq (\log\log x)^{1-\epsilon}$, we have $\left(C\cdot\frac{1}{3}u\log\log x\right)^{\frac{1}{3}u} < (\log\log x)^{\frac{2}{3}u} = y^{\frac{2}{3}}$. Now we sum on d. Using that $\frac{d/\gcd(d,\sigma(d))}{\varphi(d/\gcd(d,\sigma(d)))} \ll \log\log x = y^{o(1)}$, we find that

$$\sum_{\substack{d>y\\ \text{squarefree}}}\frac{1}{d\cdot\varphi(d/\gcd(d,\sigma(d)))} \leq y^{o(1)}\sum_{d>y}\frac{\gcd(d,\sigma(d))}{d^2} = y^{-1+o(1)},$$

where the final sum on d was estimated using Lemma 8 and partial summation. So this case contributes at most $xy^{-1/3+o(1)}$ values of n. Since $y = (\log\log x)^u > u^u$, this is acceptable.

Now suppose that $w > \frac{1}{3}u$. We can assume that d (and hence also $d/\gcd(d,\sigma(d))$) is y-smooth. Indeed, a now familiar argument shows that the number of squarefree $n \leq x$ with $\gcd(n,\sigma(n))$

divisible by a prime exceeding y is is $\ll x \log\log x/y$, which is $x/y^{1+o(1)}$ (remembering that $\log\log x = y^{1/u}$) and thus acceptable.

Viewing $d' = d/\gcd(d,\sigma(d))$, the number of remaining n can be bounded, in terms of an arbitrary parameter $t > 1$, by

$$x \sum_{\substack{d' \text{ squarefree} \\ y\text{-smooth} \\ \omega(d') > \frac{1}{3}u}} \frac{1}{d'} \le x t^{-u/3} \sum_{\substack{d' \text{ squarefree} \\ y\text{-smooth}}} \frac{t^{\omega(d')}}{d'} = x t^{-u/3} \prod_{p \le y} \left(1 + \frac{t}{p}\right)$$

$$\le x \exp\left(-\frac{u}{3}\log t + t \log\log y + O(t)\right). \tag{15}$$

We choose $t = u/\log u$. Observe that $\log\log y = \log u + \log\log\log\log x < 3\log u$ (in this range of y), so that

$$\frac{u \log t}{t \log\log y} \gtrsim \frac{(\log u)^2}{\log\log y} \gtrsim \frac{1}{3}\log u,$$

which tends to infinity. Thus, $t \log\log y = o(u \log t)$, rendering the upper bound in (15) of size at most $x\exp(-(\frac{1}{3} + o(1))u\log u)$. This completes the proof of (14).

To transition from E^* to E, let n be any integer in $[1, x]$ with $\gcd(n, \sigma(n)) > y$, and write $n = n_0 n_1$, where n_0 is squarefree, n_1 is squarefull, and $\gcd(n_0, n_1) = 1$. Then (for x large)

$$\begin{aligned} y < \gcd(n,\sigma(n)) &\le \gcd(n_0,\sigma(n_0))\gcd(n_0,\sigma(n_1))\gcd(n_1,\sigma(n_0)\sigma(n_1)) \\ &\le \gcd(n_0,\sigma(n_0))\sigma(n_1)n_1 \le 2\gcd(n_0,\sigma(n_0))n_1^2\log\log x. \end{aligned} \tag{16}$$

Thus, either (a) $n_1 > y^{2/7}/\log\log x$ or (b) $n_1 \le y^{2/7}/\log\log x$ and $\gcd(n_0,\sigma(n_0)) > y^{3/7}$. The number of $n \le x$ in case (a) is $O(x(\log\log x)^{1/2}/y^{1/7})$, which is at most $x/y^{1/7+o(1)}$ and acceptable for (3). The number of n in case (b) is bounded by

$$\sum_{\substack{n_1 \le y^{2/7}, \\ n_1 \text{ squarefull}}} E^*(x/n_1, y^{3/7}).$$

As $\log(y^{3/7})/\log\log(x/n_1) \sim \frac{3}{7}u$, our bounds on E^* yield

$$E^*(x/n_1, y^{3/7}) \le \frac{x}{n_1}\exp(-(1/7 + o(1))u\log u),$$

uniformly for squarefull $n_1 \le y^{2/7}$. Since $\sum 1/n_1 \ll 1$, the upper bound in (3) follows.

Remark. A more elaborate version of the argument going from E^* to E would lead to an improvement (increase) of the constant 1/7 in (4). We have not pursued this, since we suspect that (14) itself is not optimal.

3.4. **Upper bound in Theorem 3.** We first argue that the bound (14) can be improved to

$$E^*(x,y) \le x\exp\left(-\left(1 + o(1)\right)u\log u\right) \tag{17}$$

when $y \le \exp((\log\log x)^{o(1)})$. (Of course, we continue to assume that $u = \log y/\log\log\log x \to \infty$.) We will suppose that $u > (\log\log\log x)^{1/2}$, as our earlier arguments already establish (17) in the complementary range.

We treat this range of u by the same the method used in proving Theorem 4. However, instead of splitting the possible values of $w = \omega(d/\gcd(d, \sigma(d)))$ at $\frac{1}{3}u$, we split at $(1-\eta)u$, where $\eta > 0$ is small and fixed. Keeping in mind that $u \le (\log\log x)^{o(1)}$, we see that $(C(1-\eta)u \log\log x)^{(1-\eta)u} \le y^{1-\eta+o(1)}$, and then that the number of n corresponding to some $w \le (1-\eta)u$ is at most $xy^{-\eta+o(1)}$. Since $u = (\log\log x)^{o(1)}$, eventually $y^{-\eta+o(1)} < y^{-\eta/2} = (\log\log x)^{-u\eta/2} < \exp(-u\log u)$, so that there are fewer than $x\exp(-u\log u)$ values of n of this kind. On the other hand, there are at most $x\exp(-(1-\eta+o(1))u\log u)$ numbers n corresponding to some $w \ge (1-\eta)u$. This proves (17) with $1+o(1)$ replaced by $1-\eta+o(1)$. Since η can be taken arbitrarily small, (17) follows.

Now suppose that $n \le x$ is not necessarily squarefree and that $\gcd(n, \sigma(n)) > y$. Again, we fix a small real number $\eta > 0$.

Write $n = n_0 n_1$ where n_0 is squarefree, n_1 is squarefull, and $\gcd(n_0, n_1) = 1$. We can assume $n_1 \le y^\eta/\log\log x$, since the number of $n \le x$ with squarefull component exceeding $y^\eta/\log\log x$ is $O(x(\log\log x)^{1/2}/y^{\eta/2})$ and (eventually) $y^{\eta/2}/(\log\log x)^{1/2} > y^{\eta/3} = (\log\log x)^{u\eta/3} > u^u$. So from (16),

$$d_0 := \gcd(n_0, \sigma(n_0)) > y^{1-2\eta}.$$

Thus, given n_1, the number of corresponding n_0 is at most

$$E^*(x/n_1, y^{1-2\eta}) \le \frac{x}{n_1}\exp((1-2\eta+o(1))u\log u),$$

uniformly in $n_1 \le y^\eta$. Summing on squarefull n_1, and keeping mind that η can be taken arbitrarily small, we obtain the upper bound on $E(x,y)$ claimed in Theorem 3.

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