The parity of the multiplicative partition function

Paul Pollack

University of Illinois at Urbana-Champaign

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A conjecture of Parkin and Shanks

Let $p(n)$ be the number of partitions of n, where a partition of n is a way of writing n as a sum of natural numbers, where the order of the summands does not matter. For example, $p(5) = 7$, corresponding to

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+1+1+1, \quad 2+2+1, \quad 1+1+1+1+1.$$

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We know quite a bit about the **asymptotic properties** of $p(n)$. For example, Hardy and Ramanujan proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad (n \to \infty).$$

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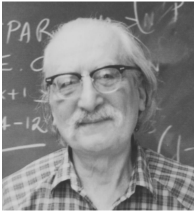
Arithmetic properties of $p(n)$ remain more mysterious, although we know much more than we used to.

Conjecture (Parkin–Shanks)

As $x \to \infty$, the values $p(n)$ become uniformly distributed modulo 2. In other words,

$$\#\{n \le x : p(n) \text{ even}\} \sim \frac{1}{2}x \quad (x \to \infty).$$



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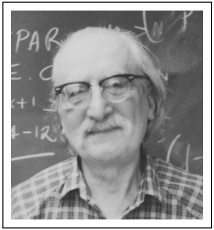

This conjecture has been attacked by several authors (Kolberg, Subbarao, Nicolas–Rusza–Sarkőzy, Ahlgren, Ono).

Theorem

For large x, we have

$$\#\{n \leq x : p(n) \text{ even}\} \gg x^{1/2}(\log\log x)^{1/2}$$

and for every fixed K,

$$\#\{n \leq x : p(n) \text{ odd}\} \gg x^{1/2}(\log\log x)^{K} / \log x.$$

Multiplicative partitions

Let $f(n)$ be the number of factorizations of n, where a *factorization* of n is a way of writing n as a product of integers all larger than 1. We consider two factorizations the same if they differ only in the order of the factors. For example, $f(12) = 4$, corresponding to

$$2 \cdot 2 \cdot 3, \quad 2 \cdot 6, \quad 3 \cdot 4, \quad 12.$$

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Again we have good asymptotic results.

Theorem (Oppenheim, Szekeres–Turán)

As $x \to \infty$,

$$\frac{1}{x} \sum_{n \le x} f(n) \sim \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}.$$

Theorem (Canfield–Erdős–Pomerance)

Let

$$L(x) := x^{\log\log\log x/\log\log x}.$$

For each fixed $\epsilon > 0$*, there are infinitely many* n *with*

$$f(n) > n/L(n)^{1+\epsilon}.$$

However, there are only finitely many n *with*

$$f(n) > n/L(n).$$

Conjecture

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Theorem (Zaharescu–Zaki)

For each $\epsilon > 0$ and all large x, we have

$$\#\{n \le x : f(n) \text{ even}\} > \left(\frac{1}{2\pi^2} - \epsilon\right) x$$

and

$$\#\{n \le x : f(n) \text{ odd}\} > \left(\frac{2}{\pi^2} - \epsilon\right) x.$$

Theorem (P.)

Fix an arithmetic progression $a \bmod m$. Then the set of n for which

$$f(n) \equiv a \pmod{m}$$

possesses an asymptotic density; that is,

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Theorem (P.)

In the case when $m = 2$ *and* $a = 1$*, this density is about* 57.1%*. So the values* $f(n)$ *are* **not** *uniformly distributed modulo* 2*.*

Revisiting the theorem of Zaharescu and Zaki

Define the kth Bell number B_k as the number of set partitions of a k-element set. Alternatively, the B_k are described by the exponential generating function

$$e^{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Theorem (Touchard, Radoux, Lunnon–Pleasants–Stephens)

The Bell numbers B_k are purely periodic to every modulus. The length of the period modulo p always divides $\frac{p^p-1}{p-1}$.

Now suppose that n is squarefree. The set of such n has a density, which is given by the product

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

For squarefree n with $k = \omega(n)$ prime factors,

$$f(n) = B_k \quad (k\text{th Bell number}).$$

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The Bell numbers start off

$$B_0 = 1, \quad B_1 = 1, \quad B_2 = 2, \quad \ldots$$

and are purely periodic modulo 2 with period $\frac{2^2-1}{2-1} = 3$. Hence, we see that the parity of f is a function of k mod 3:

$$f(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } k \equiv 0, 1 \pmod 3, \\ 0 \pmod 2 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Lemma

The values $\omega(n)$ are uniformly distributed to every modulus M, as n ranges over the squarefree numbers. (In particular, for $M = 3$.)

Proof.

It's enough to show that for each Mth root of unity $\zeta \neq 1$, the sum $\sum_{\substack{n \leq x \\ n \text{ squarefree}}} \zeta^{\omega(n)}$ possesses cancelation (is $o(x)$, as $x \to \infty$). This follows from known results on mean values of multiplicative functions.

Corollary

The density of squarefree numbers with $f(n)$ odd is $\frac{2}{3}\frac{6}{\pi^2} = \frac{4}{\pi^2}$ and the density of squarefree numbers with $f(n)$ even is $\frac{1}{3}\frac{6}{\pi^2} = \frac{2}{\pi^2}$.

The existence of the density

We want that $\mathcal{S} := \{n : f(n) \equiv 1 \pmod 2\}$ has a density. Say that a number N is *squarefull* if p^2 divides N whenever p divides N.

For each n, write $n = AB$, with A squarefull, B squarefree, and $\gcd(A, B) = 1$. Here A is called the *squarefull part* of n. For each squarefull number A, put

$$\mathcal{S}_A := \{n : f(n) \equiv 1 \pmod 2, \quad n \text{ has squarefull part } A\}.$$

Then $\mathcal{S} = \cup_A \mathcal{S}_A$.

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For n with squarefull part A, write

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Lemma (P.)

$$f(Ap_1 \cdots p_k) = \sum_{j=0}^{k} S(k,j) \sum_{d|A} f(d)\tau_j(A/d).$$

Here $S(k,j)$ is the number of set partitions of a k-element set into j parts (*Stirling number of the second kind*), and

$$\tau_j(n) = \sum_{d_1 \cdots d_j = n} 1.$$

Example (Disproof of the 50-50 conjecture)

With $A = 1$, we have $1/3$ of the time $f(n)$ is even, and $2/3$ of the time, $f(n)$ is odd.

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Collecting, we find the proportion of the time $f(n)$ is odd is at least

$$\frac{2}{3}\frac{6}{\pi^2} + \frac{1}{3}\left(\frac{6}{\pi^2}\sum_p \frac{1}{p(p+1)}\right) + \frac{2}{3}\left(\frac{6}{\pi^2}\sum_p \frac{1}{p^2(p+1)}\right) = 0.52165\ldots.$$

Thank you!