Some problems of Erdős on the sum-of-divisors function

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Dramatis Personae

Let $s(n) := \sum_{d|n, d<n} d$ denote the sum of the proper divisors of n.
So if $\sigma(n) = \sum_{d|n} d$ is the usual sum-of-divisors function, then

$$s(n) = \sigma(n) - n.$$

For example,

$$s(4) = 1 + 2 = 3, \qquad \sigma(4) = 1 + 2 + 4 = 7.$$

The ancient Greeks said that n was ...
deficient if $s(n) < n$, for instance $n = 5$;
abundant if $s(n) > n$, for instance $n = 12$;
perfect if $s(n) = n$, for example $n = 6$.

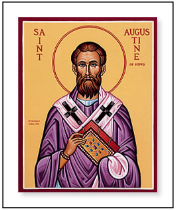
Nicomachus (60-120 AD) and the Goldilox theory

The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.

. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.

Iamblichus (245-325) and St. Augustine (354-430) on perfect numbers

The number Six .... which is said to be perfect ... was called Marriage by the Pythagoreans, because it is produced from the intermixing of the first meeting of male and female; and for the same reason this number is called Holy and represents Beauty, because of the richness of its proportions.



Six is a number perfect in itself, and not because God created all things in six days; rather, the converse is true. God created all things in six days because the number is perfect.

A deep thought

We tend to scoff at the beliefs of the ancients.

A deep thought

We tend to scoff at the beliefs of the ancients.
But we can't scoff at them personally, to their faces, and this is what annoys me.

– Jack Handey

From numerology to number theory

Perfect numbers are solutions to the equation $\sigma(N) = 2N$. What do these solutions look like?

Theorem (Euclid)

If $2^n - 1$ is a prime number, then $N := 2^{n-1}(2^n - 1)$ is a perfect number.

For example, $2^2 - 1$ is prime, so $N = 2 \cdot (2^2 - 1) = 6$ is perfect.
A slightly larger example ($\approx$ 35 million digits) corresponds to $n = 57885161$.

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Theorem (Euler)

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Problem

*Are there any **odd** perfect numbers?*

Anatomy of an odd perfect integer

. . . a prolonged meditation has satisfied me that the existence of [an odd perfect number] - its escape, so to say, from the complex web of conditions which hem it in on all sides - would be little short of a miracle.

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If N is an odd perfect number, then:

1. N has the form $p^e M^2$, where $p \equiv e \equiv 1 \pmod 4$ (Euler),
2. N has at least 10 distinct prime factors (Nielsen, 2014) and at least 101 prime factors counted with multiplicity (Ochem and Rao, 2012),
3. $N > 10^{1500}$ (Ochem and Rao, 2012).

Conjecture

There are no odd perfect numbers.

Counting perfects

Let $V'(x)$ denote the number of odd perfect numbers $n \le x$.

Theorem (Hornfeck)

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Proof.

Each odd perfect N has the form $p^e M^2$. If $N \leq x$, then $M \leq \sqrt{x}$.
We will show that each M corresponds to at most one N.
In fact, since $\sigma(p^e)\sigma(M^2) = \sigma(N) = 2N = 2p^e M^2$, we get

$$\frac{\sigma(p^e)}{p^e} = \frac{2M^2}{\sigma(M^2)}.$$

The right-hand fraction depends only on M.
The left-hand side is already a reduced fraction, since
$p \nmid 1 + p + \cdots + p^e = \sigma(p^e)$. Thus, p^e depends only on M.

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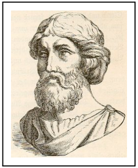
Best result: $V(x) \le x^{c/\log\log x}$ (Wirsing, 1959).

Did Pythagoras invent arithmetic dynamics?

Consider the map $s\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, extended to have $s(0) = 0$. A perfect number is nothing other than a positive integer fixed point.

We say n is **amicable** if n generates a two-cycle: in other words, $s(n) \neq n$ and $s(s(n)) = n$. For example,

$$s(220) = 284, \quad \text{and} \quad s(284) = 220.$$



Pythagoras, when asked what a friend was, replied:

One who is the other I, such are 220 and 284.

The distribution of amicable numbers

There are over ten million amicable pairs known, but we have no proof that there are infinitely many.
But we can still guess!

Let $A(x)$ be the number of pairs with smaller member $\leq x$.

Conjecture (Bratley–Lunnon–McKay, 1970)

$A(x) = o(\sqrt{x})$.

They based this on a complete list of amicable pairs to 10^7.

A voice of dissent

Here is data up to 10^{13} from a more recent survey of Garcia, Pedersen, and te Riele (2004):

x	$A(x)$	$A(x)\ln(x)/\sqrt{x}$	$A(x)\ln^2(x)/\sqrt{x}$	$A(x)\ln^3(x)/\sqrt{x}$	$A(x)\ln^4(x)/\sqrt{x}$
10^5	13	0.473	5.45	62.7	722
10^6	42	0.580	8.02	111	1530
10^7	108	0.550	8.87	143	2305
10^8	236	0.435	8.01	148	2717
10^9	586	0.384	7.96	165	3418
10^{10}	1427	0.329	7.57	174	4011
10^{11}	3340	0.268	6.78	172	4347
10^{12}	7642	0.211	5.83	161	4454
10^{13}	17519	0.166	4.96	149	4448

In contrast with B-L-McK, Erdős suggests that for each $\epsilon > 0$ and each positive integer K, one has

$$x^{1-\epsilon} < A(x) < x/(\log x)^K.$$

Upper bounds

Let $V_2(x)$ denote the number of $n \le x$ belonging to some amicable pair. (Thus, $A(x) \le V(x) \le 2A(x)$.)

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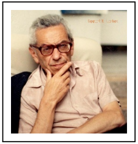
Sociable numbers

More generally, we call n a k-**sociable number** if n starts a cycle of length k. (So perfect corresponds to $k = 1$, amicable to $k = 2$.) For example,

$$2115324 \mapsto 3317740 \mapsto 3649556 \mapsto 2797612 \mapsto 2115324 \mapsto \dots$$

is a sociable 4-cycle. We know 221 cycles of order > 2.

Let $V_k(x)$ denote the number of k-sociable numbers $n \leq x$.



Theorem (Erdős, 1976)

Fix k. The set of k-sociable numbers has asymptotic density zero. In other words, $V_k(x)/x \to 0$ as $x \to \infty$.

Counting sociables

How fast does $V_k(x)/x \to 0$? Erdős's proof gives ...

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$$V_k(x)/x \leq 1/\overbrace{\log\log\cdots\log}^{3k \text{ times}} x.$$

In joint work with Mits Kobayashi and Carl Pomerance, we obtain more reasonable bounds. A further improvement is possible for odd k.

Theorem (P., 2010)

Suppose k is odd, and let $\epsilon > 0$. Then

$$V_k(x) \leq x/(\log x)^{1-\epsilon}$$

for all large x.

Dear Carl,

Many thanks for your letter + nice proof of the upper bound for amicable numbers. Put $s_1(n) = \sigma(n) - n$. Consider the integers for which $s_k(n) = n$. Can you get a good estimation for the number of these integers $n < x$. I think this can be done for fixed k but ~~a~~ it may be difficult to prove that the ~~num~~ density of integers n for which ~~σ_k~~ $s_k(n) = n$ for some k is 0.

Counting sociables

What if we count all sociable numbers at once? Put

$$V(x) := V_1(x) + V_2(x) + V_3(x) + \dots$$

Is it still true that most numbers are not sociable numbers?

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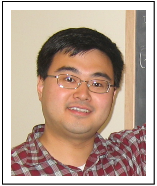
Theorem (K.–P.–P., 2009)

$$\limsup V(x)/x \leq 0.0021.$$



Theorem (K.–P.–P., 2009)

The number of $n \leq x$ which belong to a cycle entirely contained in $[1, x]$ is $o(x)$, as $x \to \infty$.

Here 0.0021 is standing in for the density of **odd abundant numbers**, odd numbers n for which $s(n) > n$ (e.g., $n = 945$).

Why do the abundant numbers have density 0?

For notational convenience, put $h(n) = \frac{\sigma(n)}{n}$. Notice that n is abundant precisely when $h(n) > 2$.

Lemma

The function h is "multiplicatively strictly increasing". In other words, if $n \mid m$, then $h(n) \leq h(m)$, and equality holds exactly when $n = m$.

Since $h(6) = 2$, if n is any multiple of 6, then $h(n) \geq 2$, and if n is a proper multiple of 6, then $h(n) > 2$.

In this way, we might say that the number 6 explains why $12, 18, 24, \ldots$ are all abundant.

Mommy, where do abundant numbers come from?

In general, whenever $h(n) \geq 2$ (that is, n is abundant or perfect), there is a *smallest* divisor d of n with $h(d) \geq 2$. We can view d as explaining the abundance of n.

The integers d that arise this way satisfy $h(d) \geq 2$ (by definition) but have no proper divisor d' with $h(d') \geq 2$. Such d are called **primitive nondeficient**.

The first several primitive nondeficient numbers are

$$6, 20, 28, 70, 88, 104, 272, 304, 368, 464, 496, 550, \ldots.$$

Theorem (Erdős, 1934)

The sum of the reciprocals of the primitive nondeficient numbers converges.

Erdős used this result to give his own proof of Davenport's theorem that the abundant numbers have an asymptotic density.

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Why is this so useful? It comes down to the following lemma.

Lemma

Let $\epsilon > 0$. There is a constant B_ϵ for which the following holds: All abundant numbers n outside of a set of upper density at most ϵ have a primitive nondeficient divisor not exceeding B_ϵ.

Thus, 99.9% of abundant numbers are "explained" by a certain finite list of primitive abundants, namely those up to B_ϵ.

Lemma

Let $\epsilon > 0$. There is a constant B_ϵ for which the following holds: All abundant numbers n outside of a set of upper density at most ϵ have a primitive nondeficient divisor not exceeding B_ϵ.

Given Erdős convergence result, the proof of the lemma is easy. The number of $n \leq x$ divisible by a given primitive nondeficient number d is $\lfloor x/d \rfloor$, and so the number divisible by such a $d > B_\epsilon$ is at most

$$x \sum_{d > B_\epsilon} \frac{1}{d}.$$

Since the sum on d converges without the restriction to $d > B_\epsilon$, we can make it smaller than ϵ by adding the condition $d > B_\epsilon$.

More puzzle pieces

Underlying many of Erdős's results is the observation that, for most n, the quantity $\sigma(n)$ is divisible by almost all small integers.

Lemma

Fix a positive integer d. For asymptotically 100% of integers n, the integer $\sigma(n)$ is divisible by d.

This is proved by analyzing the product formula for $\sigma(n)$:
If there is a prime p dividing n for which $d \mid p+1$, and $p^2 \nmid n$, then $d \mid p+1 \mid \sigma(n)$. And most of the time, there is such a prime p (the upper bound sieve).

We can now explain why amicable numbers have to be rare.

It's enough to explain why asymptotically 0% of the positive integers are the smaller member of such a pair.

Take any such number n, and say n is amicably paired with m. Then $s(n) = m > n$. Thus, n is abundant. On the other hand, $s(m) = n < m$, and so $m = s(n)$ is deficient.

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The set of n with n abundant but $s(n)$ deficient has asymptotic density 0.

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Proof (Pomerance).

In this sketch, let's settle for upper density at most 0.1%. Then we can assume n is divisible by some primitive nondeficient number $d \leq B_\epsilon$, where $\epsilon = 0.001$.

Since $s(n)$ is deficient, it cannot be divisible by d.

But $s(n) = \sigma(n) - n$. So d does not divide $\sigma(n) - n$. Since $d \mid n$, it must be that $d \nmid \sigma(n)$. This last condition puts d in a set of density 0.

Aliquot reversals

Definition

We say that n is an example of an **up-down reversal** if n is abundant but $s(n)$ is deficient. We say n is an example of a **down-up reversal** if n is deficient but $s(n)$ is abundant.

The argument we just sketched shows that the up-down reversals form a set of density 0.

It turns out to be true, but harder to prove, that the down-up reversals form a set of density 0 (Erdős–Granville–Pomerance–Spiro, 1981).

Even though one can prove that these sets have density 0, the resulting upper bounds on their counting functions are fairly large. So reversals are uncommon, but we can't show they're *that* uncommon.

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In a recent paper with Carl, we explain why: reversals **are** actually fairly common.

Theorem (Pomerance and P., 2015)

The number of up-down reversals in $[1, x]$ *is*

$$\gg \frac{x}{(\log_2 x)^2 (\log_3 x)^3}$$

while the number of down-up reversals in $[1, x]$ *is*

$$\gg \frac{x}{(\log_2 x)^2 (\log_3 x)^2}.$$

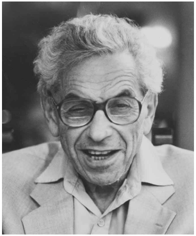
ssssssssssssssssss. . .

Saying that up-down reversals are rare is the same as saying that an abundant number, under one application of s, usually stays abundant. And similarly for deficient numbers, from the result on down-up reversals.

What about if one iterates s? If n is abundant, is $s(s(n))$ almost always abundant? If n is deficient, is $s(s(n))$ almost always deficient.

Theorem (Erdős, Granville, Pomerance, Spiro)

Fix k. For all abundant n outside of a set of density 0, all of n, $s(n)$, $\ldots$, $s_k(n)$ are abundant.

Paul Erdős, Andrew Granville, Carl Pomerance, Claudia Spiro

Conjecture

The analogous result holds with "abundant" replaced by "deficient".

This is open even for $s(s(n))$. In other words, we don't know how to show that if n is deficient, then almost always $s(s(n))$ is deficient.

The full conjecture can be shown to follow from another conjecture of EGPS, of independent interest:

Conjecture

If $\mathcal{A}$ *is a set of integers of asymptotic density* 0*, then the preimage set* $s^{-1}(\mathcal{A})$ *also has asymptotic density zero.*

Is 1 the loneliest number?

The question of the distribution of sociable numbers is a question about the fixed points of the function s.

There are other natural questions about s from this function-theoretic viewpoint. Perhaps most obvious is to ask what one can say about the range of s, i.e., the set $s(\mathbb{N})$.

Investigations here date back to at least the 10th century, specifically the work of Ibn Tahir al-Baghdadi. He referred to the input n as the "begetter" and the output $s(n)$ as the "begotten".

As to the begetter and the begotten among the numbers, so the sum of the aliquot parts of any number is the begotten of this number, which itself is the begetter of its aliquot parts. Now, 5 among the odd numbers and 2 among the even numbers have no begetter, since there is no number such that the sum of its aliquot parts be 5 or 2. Hence, they stand among the numbers like a bastard among the people.

Definition

A number not in the range of s is called *nonaliquot*.

al-Baghdadi observed that 2 and 5 are nonaliquot.

If n is composite with smallest prime divisor p, then

$$s(n) \geq 1 + n/p \geq 1 + \sqrt{n}.$$

This gives a simple algorithm to decide whether or not a number belongs to the range of s.

Here are the first several nonaliquots:

$$2, 5, 52, 88, 96, 120, 124, 146, 162, 188, 206, 210, 216, 238.$$

To explain why odds don't seem to be appearing, note that if p and q are distinct primes, then

$$s(pq) = 1 + p + q.$$

Probably every even number $n \geq 8$ is a sum of two distinct primes. Hence, every odd $n \geq 9$ is in the range of s.

Since $s(2) = 1$, $s(4) = 3$, and $s(8) = 7$, the only odd nonaliquot number is 5, provided we are happy assuming a plausible strengthening of the Goldbach conjecture.

Where does Erdős enter the picture?

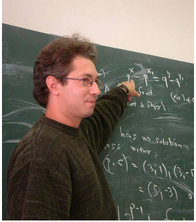
Theorem (Erdős, 1973)

A positive proportion of even numbers are untouchable.

The proof is clever but not so long (about a page); it makes essential use of the lemma we mentioned before, that for each fixed d, we have $d \mid \sigma(n)$ for almost all n.

Lower bounds on the proportion of numbers that are even and untouchable have been investigated by te Riele (1976), Banks and Luca (2005), and Chen and Zhao (2005). In this last paper, it is show that the proportion of positive integers n that are both even and untouchable is > 0.06.

Erdős's method is powerless to answer the question of whether or not a positive proportion of even numbers are in the range.



Theorem (Luca and Pomerance, 2015)

In every arithmetic progression, a positive proportion of the members belong to the range of s.

One could obtain a numerical lower bound on the density of even "touchable" numbers from the argument, but it would not be a pleasant exercise.

The truth?

Using an algorithm introduced by Yang–Pomerance, one can compute counts of untouchables to reasonably large heights.

x	$U(x)$	$D(x)$	x	$U(x)$	$D(x)$
100000000	16246940	0.1625	1000000000	165826606	0.1658
200000000	32721193	0.1636	2000000000	333261274	0.1666
300000000	49265355	0.1642	3000000000	501171681	0.1671
400000000	65855060	0.1646	4000000000	669372486	0.1673
500000000	82468000	0.1649	5000000000	837801755	0.1676
600000000	99107582	0.1652	6000000000	1006383348	0.1677
700000000	115764316	0.1654	7000000000	1175094232	0.1679
800000000	132438792	0.1655	8000000000	1343935989	0.1680
900000000	149128373	0.1657	9000000000	1512867678	0.1681
			10000000000	1681871718	0.1682

Guess: $\approx 17\%$ of natural numbers are even and untouchable.

Conjecture (Pomerance and P., 2015)

The set of nonaliquot numbers has asymptotic density Δ, where

$$\Delta = \lim_{y\to\infty} \frac{1}{\log y} \sum_{\substack{a\le y\\ 2|a}} \frac{1}{a} \mathrm{e}^{-a/s(a)}.$$

The constant Δ is

- well-defined (i.e., we can prove the limit exists)
- effective computable

Plugging in $y = 2\cdot 10^{10}$, we get an approximation to Δ of 0.171822.

I get by with a little help ...

Definition

Distinct natural numbers n, m are called **friends** if $\frac{\sigma(n)}{n} = \frac{\sigma(m)}{m}$.

Example

The friends of 6 are $28, 496, 8128, \ldots$ — i.e., the other perfect numbers.

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The number 5 has no friends. Notice that $\frac{\sigma(5)}{5} = 6/5$. If $\frac{\sigma(n)}{n} = \frac{6}{5}$, then $5 \mid n$. But $\frac{\sigma(n)}{n} > \frac{\sigma(5)}{5}$ unless $n = 5$. This argument clearly works for every prime number n. In fact, it works whenever n and $\gcd(n, \sigma(n)) = 1$.

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If n has no friends, we say n is **solitary**. Thus, n is solitary whenever $\gcd(n, \sigma(n)) = 1$.

Example

We do not know if $n = 10$ is solitary or not. In other words, we do not know if there is any solution to $\frac{\sigma(n)}{n} = \frac{9}{5}$ except $n = 10$ itself.

There is no known algorithm to decide whether or not a given integer n has a friend.

Perhaps not surprisingly, we do not have a good estimate on the number of $n \leq x$ which are solitary. The $(n, \sigma(n)) = 1$ criteria implies (by work of Erdős) that there are at least

$$(e^{-\gamma} + o(1))x / \log\log\log x$$

such numbers $n \leq x$.

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such numbers $n \leq x$.

Problem

Prove or disprove that a positive proportion of natural numbers are solitary.

If you can't answer a question... answer a different question.

To make life easier, let's restrict attention entirely to an initial interval $[1, x]$. We say that a number $n \in [1, x]$ is x-solitary if it has no friends in $[1, x]$.

Question

Is there a limiting proportion of x-solitary numbers, as $x \to \infty$? And is this proportion positive?

Note that the answer to the last question can be positive without the proportion of solitary numbers being positive.

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We can answer these questions in the affirmative. In fact, we can show more.

For each integer $k \geq 0$, let $N(x; k)$ be the number of positive integers $n \leq x$ with at least k friends in $[1, x]$.

Theorem (Pomerance and P., 2015)

For each fixed $k \geq 0$,

$$\frac{N(x; k)}{x} \to \alpha_k \quad (\textit{as } x \to \infty)$$

for some α_k.

Clearly, $1 = \alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \dots$.

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Theorem

If there is a single example of an integer with at least k friends, then $\alpha_k > \alpha_{k+1}$.

In particular, the limiting proportion of x-solitary numbers in $[1, x]$ is $\alpha_0 - \alpha_1$, and this is positive.

x	10^6	10^7	10^8	10^9	10^{10}
$N_1(x)$	34746	347471	3474564	34745605	347456117
$N_2(x)$	2816	28089	280938	2809813	28097701
$N_3(x)$	857	8517	85365	853513	8535154
$N_4(x)$	85	853	8457	84605	845674

Perhaps $\alpha_1 = 0.0347\ldots$, $\alpha_2 = 0.0028\ldots$, $\alpha_3 = 0.00085\ldots$, and $\alpha_4 = 0.000084\ldots$.


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