Summing divisors: A report on the first two thousand years

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The quest for perfection

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But who decided adding divisors was a reasonable thing to do in the first place?

All Greek to us

> *Among simple even numbers, some are superabundant, others are deficient: these two classes are as two extremes opposed one to the other; as for those that occupy the middle point between the two, they are said to be perfect.*
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> – Nicomachus (ca. 100 AD), *Introductio Arithmetica*

Abundant: $s(n) > n$, e.g., $n = 12$.
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Carl Pomerance has called this the "Goldilox classification".

Goldilox explained

The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.

. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.

You can see a lot just by looking

Let's list the first several terms of each of these sequences.

Abundants: 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, ...

Deficients: 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, ...

Perfects: 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, ...

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> *Just as . . . ugly and vile things abound, so superabundant and deficient numbers are plentiful and can be found without a rule. . . – Nicomachus*

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If A is a subset of $\mathbb{N} = \{1, 2, 3, \dots\}$, define the *density* of A as

$$\lim_{x\to\infty} \frac{\#A \cap [1, x]}{x}.$$

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Question

Does the set of abundant numbers have a density? What about the deficient numbers? The perfect numbers?

It's OK to be dense, ctd.

Theorem (Davenport, 1933)

For each real $u \geq 0$, consider the set

$$\mathcal{D}_s(u) = \{n : s(n)/n \leq u\}.$$

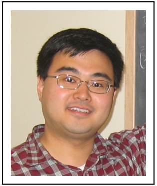
This set always possesses an asymptotic density $D_s(u)$. Considered as a function of u, the function D_s is continuous and strictly increasing, with $D_s(0) = 0$ and $D_s(\infty) = 1$.

Corollary

The perfect numbers have density 0, the deficient numbers have density $D_s(1)$, and the abundant numbers have density $1 - D_s(1)$.

Numerics

The following theorem improves on earlier work of Behrend, Salié, Wall, and Deléglise:



Theorem (Kobayashi, 2010)

For the density of abundant numbers, we have

$$0.24761 < 1 - D_s(1) < 0.24765.$$

So just under 1 *in every* 4 *natural numbers is abundant, and just over* 3 *in* 4 *are deficient.*

Interlude: Large values of $\frac{s(n)}{n}$

According to Davenport's theorem, $D_s(u) < 1$ for all u. In other words, no matter how large u is, a positive proportion of numbers n have

$$\frac{s(n)}{n} > u.$$

Can we see why this should be the case?

It will be convenient in what follows to work not with $\frac{s(n)}{n}$ but with $\frac{\sigma(n)}{n}$. Since $\sigma(n) = n + s(n)$, we have $\frac{\sigma(n)}{n} = 1 + \frac{s(n)}{n}$.

Proposition

For each natural number n,

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d}.$$

For example, if $n = 6$, the right-hand sum is

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{6}(6+3+2+1) = \frac{1}{6}\sigma(6).$$

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In general,

$$\frac{\sigma(n)}{n} = \frac{1}{n}\sum_{d|n} d = \sum_{d|n} \frac{d}{n} = \sum_{d|n} \frac{1}{n/d}.$$

But as d runs over the divisors of n from smallest to largest, n/d also runs over the divisors of n, but in the reverse order.

Proposition

For each natural number n,

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d}.$$

Corollary

If $n \mid m$*, then* $\frac{\sigma(n)}{n} \leq \frac{\sigma(m)}{m}$*, with equality only when* $m = n$*.*

Now we produce a positive proportion of numbers n with $\sigma(n)/n > u$, for any given starting number u.

Consider $n = N!$, where N is a large positive integer.

Then

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{N}.$$

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This gives us one value of n with $\frac{\sigma(n)}{n} > u$. In fact, if n is any multiple of $N!$, then $\frac{\sigma(n)}{n} \geq \frac{\sigma(N!)}{N!} \geq u$. And the multiples of N have positive density.

It's a short jump from $\frac{\sigma(N)}{N}$ can be arbitrarily large to a beautiful theorem of Leonhard Euler:

Theorem

$$\sum_{p \text{ prime}} \frac{1}{p} \textit{ diverges.}$$

Note in particular that this implies there are infinitely many primes!

Indeed, for any number $n > 1$, one can factor $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, and then

$$
\begin{aligned}
\frac{\sigma(n)}{n} &= \sum_{d|n} \frac{1}{d} \\
&= (1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \cdots + \frac{1}{p_1^{e_1}}) \cdots (1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \cdots + \frac{1}{p_k^{e_k}}) \\
&< (1 + \frac{2}{p_1}) \cdots (1 + \frac{2}{p_k})
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Using that $e^x > 1 + x$ for every real number $x > 0$ (from Taylor series),

$$\frac{\sigma(n)}{n} < e^{2/p_1 + 2/p_2 + \cdots + 2/p_k}.$$

Thus, $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} > \frac{1}{2} \log \frac{\sigma(n)}{n}$.
So $\sum_{p \text{ prime}} \frac{1}{p} > \frac{1}{2} \log \frac{\sigma(n)}{n}$ for every n.

I get by with a little help ...

Definition

Distinct natural numbers n, m are called **friends** if $\frac{\sigma(n)}{n} = \frac{\sigma(m)}{m}$.

Example

The friends of 6 are $28, 496, 8128, \ldots$ — i.e., the other perfect numbers.

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If n has no friends, we say n is **solitary**. Thus, n is solitary whenever $\gcd(n, \sigma(n)) = 1$.

Example

We do not know if $n = 10$ is solitary or not. In other words, we do not know if there is any solution to $\frac{\sigma(n)}{n} = \frac{9}{5}$ except $n = 10$ itself.

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Problem

Prove or disprove that a positive proportion of natural numbers are solitary.

The numbers with $\gcd(n, \sigma(n)) = 1$ don't quite make up a set of positive density. Erdős showed that the number of these n up to x is roughly const $\times \frac{x}{\log\log\log x}$ for large x.

Thank you!