ON DICKSON'S THEOREM CONCERNING ODD PERFECT NUMBERS

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Abstract. A 1913 theorem of Dickson asserts that for each fixed natural number k, there are only finitely many odd perfect numbers N with at most k distinct prime factors. We show that the number of such N is bounded by 4^{k^2}.

1. Introduction

If N is a natural number, we write $\sigma(N) := \sum_{d|N} d$ for the sum of the divisors of N. We call N *perfect* if $\sigma(N) = 2N$, i.e., if N is equal to the sum of its proper divisors. The even perfect numbers were completely classified by Euclid and Euler, but the odd perfect numbers remain utterly mysterious: despite millennia of effort, we don't know of a single example, but we possess no argument ruling out their existence.

In 1913, Dickson [2] proved that for each fixed natural number k, there are only finitely many odd perfect numbers N with $\omega(N) \leq k$. (Here and below, we write $\omega(N)$ for the number of distinct prime factors of the natural number N.) The first explicit bounds were given by Pomerance [7], who showed that any such N satisfies

$$N \leq (4k)^{(4k)^{2^{k^2}}}.$$

After the work of Heath-Brown [4], and its subsequent refinements by Cook [1] and Nielsen [5], we know that any such N satisfies

$$N < 2^{4^k}. \tag{1}$$

In addition to an upper bound on the *size* of such N, it is sensible to ask for a bound on the *number* of such N. The purpose of this note is to prove the following estimate:

Theorem 1. *For each positive integer k, the number of odd perfect numbers N with $\omega(N) \leq k$ is bounded by 4^{k^2}.*

It is amusing to note the typographical similarities between the bound 2^{4^k} of (1) and our (much smaller!) bound of 4^{k^2}. Theorem 1 is a corollary of the following result that is perhaps of independent interest:

Theorem 2. *Let $x \geq 1$ and let $k \geq 1$. The number of odd perfect $N \leq x$ with $\omega(N) \leq k$ is bounded by $(\log x)^k$.*

The proofs are self-contained except for the use of the bound (1) and an appeal to the following classical result of Sylvester [8]: if N is odd and perfect, then $\omega(N) \geq 5$. (For a detailed account of Sylvester's investigations into odd perfect numbers, see [3].) Recently Nielsen [6] has shown that actually $\omega(N) \geq 9$.

Most of our notation will be familiar to students of elementary number theory. A possible exception is the definition of "$\|$" (or *exactly divides*): if p is a prime, we write $p^e \parallel n$ to mean that $p^e \mid n$ while $p^{e+1} \nmid n$.

2. Proofs

Proof of Theorem 2. We employ a modification of Wirsing's method from [9]. Suppose that $N \leq x$ is odd and perfect and $\omega(N) \leq k$. (If there are no such N, then the theorem holds trivially, since the quantity $(\log x)^k$ is nonnegative.) Let p_0 be the least prime divisor of N, and let $e_0 \geq 1$ be such that $p_0^{e_0} \parallel N$. Put $B := p_0^{e_0}$ and write $N = AB$. Then A and B are relatively prime, and so (since σ is a multiplicative function)

$$2AB = \sigma(A)\sigma(B). \tag{2}$$

Thus

$$B < \sigma(B) = \frac{2A}{\sigma(A)}B \leq 2B, \tag{3}$$

with equality on the right exactly when $A = 1$. Suppose $A \neq 1$. Then the preceding inequalities show that $\sigma(B) \nmid 2B$, and so there is a prime p_1 dividing $\sigma(B)$ to a higher power than that to which it divides $2B$; for definiteness, fix p_1 as the least such prime. It now follows from (2) that $p_1 \mid A$. Let $e_1 \geq 1$ be such that $p_1^{e_1} \parallel A$. Then if we put

$$A' := A/p_1^{e_1} \quad \text{and} \quad B' := Bp_1^{e_1},$$

we have (2) with A' in place of A and B' in place of B. Reasoning as above, we find that unless $A' = 1$, there is a prime p_2 dividing $\sigma(B')$ to a higher power than that to which it divides $2B'$. Again, for definiteness, let p_2 be the least such prime. Then $p_2^{e_2} \parallel A'$ for some $e_2 \geq 1$. We put

$$A'' := A'/p_2^{e_2} \quad \text{and} \quad B'' := B'p_2^{e_2}$$

and observe that we now have (2) with A'' and B'' replacing A and B. We continue choosing primes p_i and exponents e_i in the above manner, stopping at the lth step (say) when $A^{(l)} = 1$. At that point

$$A = p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}$$

and

$$l = \omega(A) = \omega(N/B) = \omega(N) - 1.$$

By the result of Sylvester alluded to above, we have $\omega(N) \geq 5$, and so

$$4 \leq l \leq k - 1.$$

We now count the number of possibilities for A and B. Observe that

$$\sigma(N) = \prod_{p^e \parallel N} (1 + p + p^2 + \cdots + p^e) \leq N \prod_{p \mid N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right),$$

so that

$$\frac{1}{2} = \frac{N}{\sigma(N)} \geq \prod_{p \mid N} \left(1 - \frac{1}{p}\right) \geq 1 - \sum_{p \mid N} \frac{1}{p} \geq 1 - \frac{k}{p_0},$$

from which it follows that $p_0 \leq 2k$. Since p_0 is an odd prime, the number of possibilities for p_0 is bounded by $k-1$. Since $3^{e_0} \leq B = p_0^{e_0} \leq N \leq x$, we have $e_0 \leq \log x/\log 3$. So the number of possibilities for B is crudely bounded by

$$(k-1)\log x/\log 3.$$

Next, we observe that the prime p_1 depends only on B, while for $i > 1$, the prime p_i depends only on B and the exponents $e_1, \ldots, e_{i-1}$. It follows that for a given B, the cofactor A is entirely determined by the sequence of exponents $e_1, \ldots, e_l$. Since $A \leq N \leq x$, each $e_i \leq \log x/\log p_i$. Moreover, since $4 \leq l \leq k-1$ and $p_i > p_0 \geq 3$ for $i \geq 1$[1], the number of possibilities for the sequence $e_1, \ldots, e_l$ is bounded by

$$(k-4)(\log x/\log 5)^{k-1}. \tag{4}$$

Hence the number of possibilities for $N = AB$ is bounded by

$$\frac{(k-1)(k-4)}{(\log 3)(\log 5)^{k-1}}(\log x)^k.$$

By elementary calculus, the coefficient of $(\log x)^k$ in this expression is a decreasing function of k for $k \geq 8$. Moreover, for $k \in \{5, 6, 7, 8\}$, the largest value of this coefficient is $0.942719\ldots < 1$. The theorem follows. $\square$

Proof of Theorem 1. Put $x := 2^{4^k}$. By (1) and Theorem 2, the number of odd perfect N with $\omega(N) \leq k$ is at most $(\log x)^k < (4^k)^k = 4^{k^2}$. $\square$

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[1] The published version of this paper contained the erroneous assertion that "$3 \leq p_0 < p_1 < p_2 < \ldots$." Thanks to Pace Nielsen for pointing this out.

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