The distribution of sociable numbers

Paul Pollack
Illinois/IAS
(joint with Mits Kobayashi and Carl Pomerance)

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Some definitions

For each natural number n, put

$$\sigma(n) := \sum_{d|n} d, \quad s(n) := \sum_{d|n, d<n} d,$$

so that $s(n) = \sigma(n) - n$. We say that n is *perfect* if

$$\sigma(n) = 2n, \quad \text{or equivalently,} \quad s(n) = n.$$

We will need notation for the iterates of s:
Put $s_0(n) = n$, and for $k > 0$, whenever $s_{k-1}(n)$ is defined and positive, put $s_k(n) = s(s_{k-1}(n))$.

Nicomachus and his ‘Goldilocks theory’

The super*abundant number* is . . . as if an adult animal was formed from too many parts or members, . . . having ten mouths or nine lips, and provided with three lines of teeth, or with a hundred arms . . .

The *deficient number* is . . . if an animal lacked members or natural parts . . . if one of his hands has less than five fingers, or if he does not have a tongue or something like that . . .

In the case of those that are bound between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty, and things of that sort — of which the most exemplary form is that type of number which is called *perfect*.

The distribution of perfect numbers

Perfect numbers, like perfect men, are very rare. − Descartes

Theorem (Euclid–Euler). *If N is an even perfect number, then N has the form*

$$2^{p-1}(2^p - 1),$$

where $2^p - 1$ is prime.

Conjecture. *There are no odd perfect numbers.*

Let $V(x)$ be the number of perfect $n \leq x$.
We expect $V(x) \ll \log x$ or even $V(x) \ll \log\log x$.

Theorems.

Kanold, 1954	$V(x) = o(x)$
Volkmann, 1955	$= O(x^{5/6})$
Hornfeck, 1955	$= O(x^{1/2})$
Kanold, 1956	$= o(x^{1/2})$
Erdős, 1956	$= O(x^{1/2-\delta})$
Kanold, 1957	$= O\left(x^{1/4}\frac{\log x}{\log\log x}\right)$
Hornfeck & Wirsing, 1957	$= O(x^{\epsilon})$

Best result is due to Wirsing (1959): $V(x) < x^{c/\log\log x}$.

Perfect numbers and their friends

Two natural numbers n and m are said to form an *amicable pair* if $s(n) = m$ and $s(m) = n$. In this case we say that both n and m are *amicable numbers*.

Example: 220 and 284 form an amicable pair, since

$$s(220) = 1+2+4+5+10+11+20+22+44+55+110 = 284,$$

$$s(284) = 1 + 2 + 4 + 71 + 142 = 220.$$

. . . these numbers have a particular influence in establishing union and friendship between individuals. – Ibn Khaldun

We know over a million amicable pairs, but we don't know how to prove that there are infinitely many.

Let $V_2(x)$ be the number of amicable numbers $n \le x$.

We don't know that $V_2(x) \to \infty$, but we think it does.
But we don't know how fast!

Conjecture (Bratley, Lunnon, and McKay). *As* $x \to \infty$,

$$V_2(x) = o(x^{1/2}).$$

Conjecture (Erdős). *For each* $\epsilon > 0$ *and all* $x > x_0(\epsilon)$,

$$V_2(x) > x^{1-\epsilon}.$$

What can we prove about amicable numbers?

Theorem (Erdős, 1955). *As* $x \to \infty$,

$$V_2(x) = o(x).$$

This was improved by Rieger (1973), Erdős and Rieger (1975), and Pomerance (1976). The modern record is:

Theorem (Pomerance, 1981). *For large* x,

$$V_2(x) < x/\exp((\log x)^{1/3}).$$

A sketch of the proof that almost all numbers are not amicable

Take a natural number $n \le x$, and write

$$s(n) = \sigma(n) - n.$$

For all small primes p (say $p \lessapprox \log\log x$) and almost all n, the number $\sigma(n)$ is divisible by a high power of p.

Consequence: For most numbers n,

$$v_p(s(n)) = v_p(n)$$

for all small primes p.

But

$$\frac{\sigma(n)}{n} = \prod_{p^e \| n} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^e}\right)$$

is mostly influenced by the small primes. So we expect

$$\sigma(s(n))/s(n) \approx \sigma(n)/n.$$

Write $s_k(n)$ for the kth iterate of s applied to n (if it exists). This idea can be extended to prove:

Theorem (Erdős). *Fix $K \geq 1$. Away from a set of density zero, if n is abundant, then so are all of $s(n), ..., s_K(n)$.*

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Proof that the amicable numbers have density zero: If n and m form an amicable pair with $n < m$, then

$$s(n) = m > n.$$

So n is abundant. But $m = s(n)$ is deficient, since

$$s(m) = n < m.$$

So by Erdős's theorem, n belongs to a set of density zero. □

Sociable numbers

Say that a number n is k-*sociable* if $s_k(n) = n$ and $s_j(n) \neq n$ for $1 \leq j < k$. A number is *sociable* if it is k-sociable for some $k \geq 1$.

Notice: *perfect* = 1-sociable, and *amicable* = 2-sociable.

- For a fixed k, what can be said about the function $V_k(x)$ defined by

$$V_k(x) = \#\{n \leq x : n \text{ is } k\text{-sociable}\} \quad ?$$

- What about the counting function $V^*(x)$ of *all* sociable numbers, defined by

$$V^*(x) := V_1(x) + V_2(x) + V_3(x) + \dots \quad ?$$

k-sociable numbers for fixed k

Erdős's theorem immediately implies the following result:
Theorem (Erdős). *For each fixed k, we have $V_k(x) = o(x)$.*

Erdős's argument gives

$$V_k(x) \ll_k \frac{x}{\log\log\log\cdots\log x},$$

where the denominator is a $(3k)$-fold logarithm.

For fixed $k > 2$, the best known results are:

$$V_k(x) \le \begin{cases} x/\exp((1+o(1))\sqrt{\log_3 x \log_4 x}) & \text{if } 2 \mid k, \\ x/(\log x)^{1+o(1)} & \text{if } 2 \nmid k. \end{cases}$$

The behavior of $V^*(x)$

Conjecture. *We have $V^*(x) = o(x)$. In other words, the set of sociable numbers has density zero.*

Theorem (Kobayashi, P., and Pomerance)**.** *The number sociable numbers whose cycle is contained entirely in $[1, x]$ is $o(x)$.*

Theorem (Kobayashi, P., and Pomerance)**.** *The set of sociable numbers which are not both odd and abundant has density zero.*

The odd abundant numbers have density $\approx 1/500$, so we are 99.8% of the way to a proof of the conjecture!