HOW OFTEN IS EULER'S TOTIENT A PERFECT POWER?

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Abstract. Fix an integer $k \geq 2$. We investigate the number of $n \leq x$ for which $\varphi(n)$ is a perfect kth power. If we assume plausible conjectures on the distribution of smooth shifted primes, then the count of such n is at least $x/L(x)^{1+o(1)}$, as $x \to \infty$, where $L(x) = \exp(\log x \cdot \log\log\log x/\log\log x)$. This lower bound is implicit in work of Banks–Friedlander–Pomerance–Shparlinski. We prove — unconditionally — that $x/L(x)^{1+o(1)}$ serves as an upper bound. In fact, we establish this same bound for the count of $n \leq x$ for which $\varphi(n)$ is squarefull. The proof builds on methods recently introduced by the author to study "popular subsets" for Euler's function.

1. Introduction

Let $\varphi(n)$ denote Euler's totient function, i.e., $\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$. One of several problems studied in the paper [2] of Banks, Friedlander, Pomerance, and Shparlinski is the question of how often $\varphi(n)$ assumes square values.

The methods of [2] for obtaining lower bounds in this problem depend on the distribution of 'smooth shifted primes'. Recall that a positive integer n is called *Y-smooth* (alternatively, *Y-friable*) if there are no primes dividing n that exceed Y. We write $\Psi(X,Y)$ for the count of Y-smooth $n \leq X$. It is natural to guess that shifted primes $p-1$ are smooth with roughly the same frequency as ordinary integers of the same size. This thought, which in vague form goes back at least to Erdős's work in [4], led Pomerance to float the following precise conjecture in [10].

Conjecture 1. *If $X, Y \to \infty$ with $X \geq Y$, then*

$$\#\{p \leq X : p-1 \text{ is } Y\text{-smooth}\} \sim \frac{\Psi(X,Y)}{\log X}.$$

Conjecture 1 appears rather difficult, and may even be overly optimistic. But the authors of [2] show that even much weaker statements in this direction have interesting consequences for the distribution of square values of $\varphi(n)$. Specifically, suppose that for a certain fixed $U > 1$ and a certain real number K, we have that

$$\#\{p \leq X : p-1 \text{ is } X^{1/U}\text{-smooth}\} \gg \frac{\Psi(X, X^{1/U})}{(\log X)^K} \tag{1}$$

for all large X. Then it is shown in [2] that there are at least $x^{1-1/U+o(1)}$ values of $n \leq x$ for which $\varphi(n)$ is a square, as $x \to \infty$. Baker and Harman have proved that (1) holds for any $U \leq 3.3772$. Consequently, $\varphi(n)$ is a square for at least $x^{0.7038}$ values of $n \leq x$. Conjecture 1 would predict that any $U > 1$ is admissible in (1), so that there are more than $x^{1-\epsilon}$ values of $n \leq x$ with $\varphi(n)$ a square.

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Running these arguments with U allowed to vary, we can obtain a sharper lower bound than $x^{1-\epsilon}$. This is the substance of our first theorem, which also treats kth powers rather than merely squares. Writing $\log_j$ for the jth iterate of the natural logarithm, let

$$L(x) = \exp\left(\frac{\log x \cdot \log_3 x}{\log_2 x}\right).$$

Theorem 1 (conditional on Conjecture 1)**.** *Fix an integer $k \geq 2$. As $x \to \infty$, the number of $n \leq x$ for which $\varphi(n)$ is a kth power is at least $x/L(x)^{1+o(1)}$.*

(It will emerge from the proof that the full strength of Conjecture 1 is not needed; all we need is a lower bound of roughly the predicted size when $Y \approx \exp(\sqrt{\log X})$.)

While Theorem 1 does not appear explicitly in [2], our argument below is a straightforward modification of the proof of Theorem 6.4 in [2]. We include the proof for completeness, and for comparison with our main result, which we now turn to.

The discussion so far has focused on lower bounds. As regards upper bounds, it is proved in [2] that (as $x \to \infty$) the number of $n \leq x$ for which $\varphi(n)$ is squarefull does not exceed

$$x \exp(-(1 + o(1))\sqrt{\log x \cdot \log\log\log x}).$$

Of course, this serves as an upper bound on the number of $n \leq x$ for which $\varphi(n)$ is a kth power (for any $k \geq 2$). There is quite a gap between this upper bound and the lower bound of $x/L(x)^{1+o(1)}$ in Theorem 1. The main purpose of this note is to close this gap.

Theorem 2. *As $x \to \infty$, the number of $n \leq x$ for which $\varphi(n)$ is squarefull is at most $x/L(x)^{1+o(1)}$.*

The proof of Theorem 2 uses methods recently introduced by the author in [8] to study popular values of Euler's function (cf. [10, 11, 12]).

The cognate problem of counting squarefull m in the range of the Euler function (rather than their preimages, as done here) was recently studied in [9]. It was shown there that

$$x/(\log x)^3 \ll \#\{\text{squarefull } m \leq x^2 : m \in \varphi(\mathbb{N})\} \ll x/(\log x)^{0.0063}.$$

In fact, the lower bound holds for the number of squares up to x^2 in the range of φ. For related results, see [3, 5, 6].

We note that the expression $x/L(x)^{1+o(1)}$ also arises in the study of counterexamples to the converse of Fermat's little theorem. Conjecturally, it describes the count of Carmichael numbers up to x, as well the count of base a pseudoprimes for each fixed $a \neq 0, \pm 1$. For Carmichael numbers, the upper bound implicit in the conjecture is known, while for base a pseudoprimes, an upper bound of $x/L(x)^{1/2}$ (for $x > x_0(a)$) has been shown. The heuristic lower bound arguments invoke a variant of Conjecture 1. For details, see [11].

Notation. The letter p is reserved throughout for primes. For $Y > 0$, we define arithmetic functions $\omega_{>Y}(\cdot)$ and $\Omega_{>Y}(\cdot)$ by

$$\omega_{>Y}(n) = \sum_{\substack{p \mid n \\ p > Y}} 1, \quad \text{and} \quad \Omega_{>Y}(n) = \sum_{\substack{p^e \parallel n \\ p > Y}} e;$$

when $Y = 0$, we omit the subscripts and write simply $\omega(\cdot)$ and $\Omega(\cdot)$.

2. Proof of Theorem 1

The following result in combinatorial group theory follows from combining Theorem 1.1 and Proposition 1.2 of [1].

Lemma 3. *Let G be a finite abelian group of exponent m, and let*

$$d = \lceil m(1 + \log(|G|/m)) \rceil.$$

For each pair of integers (r, t) with $r > t > d$, any sequence of r elements of G contains at least $\binom{r}{t}/\binom{r}{d}$ distinct subsequences of length at most t and at least $t - d$, whose product is the identity element.

Proof of Theorem 1. Let $X = \exp((\log_2 x)^2)$ and $y = \log x/(\log_2 x)^3$. Let r denote the number of primes $p \le X$ for which $p - 1$ is y-smooth. We will construct the values of n in Theorem 1 as certain squarefree products of at most t of these r primes, where

$$t := \lfloor \log x/(\log_2 x)^2 \rfloor.$$

Note that all such products are at most x.

For each of the r primes p described above, write $p - 1 = \prod_{\ell \le y} \ell^{v_\ell(p-1)}$, where ℓ runs over the primes at most y. We associate to p the (mod k)-reduced exponent vector

$$(v_\ell(p-1) \bmod k)_{\ell \le y} \in (\mathbb{Z}/k\mathbb{Z})^{\pi(y)}.$$

Asking for the product n of t of these primes to have $\varphi(n)$ a kth power amounts to asking that the sum of the corresponding exponent vectors be the zero element in $G := (\mathbb{Z}/k\mathbb{Z})^{\pi(y)}$. This implies that the number of n as in Theorem 1 is at least $\binom{r}{t}/\binom{r}{d}$, where $d = d(G)$ is the quantity of Lemma 3.

Assuming Conjecture 1, we have for large x that

$$r \gg \Psi(X, y)/\log X.$$

Now applying standard estimates for $\Psi(\cdot,\cdot)$ (for example, Theorem 2.1 of [12]), we find that, with $U := \frac{\log X}{\log y}$,

$$r \gg \frac{X \exp(-(1 + o(1))U \log U)}{\log X} = X \exp(-(1 + o(1)) \log_2 x \log_3 x).$$

Since G has exponent k,

$$d \le 1 + k(1 + \log(k^{\pi(y)-1})) < y$$

(once x is large enough). Thus, $\binom{r}{d} \le r^d \le X^y = L(x)^{o(1)}$. On the other hand,

$$\binom{r}{t} \ge \left(\frac{r}{t}\right)^t = r^t t^{-t} \ge r^t L(x)^{o(1)} \ge X^t/L(x)^{1+o(1)} = x/L(x)^{1+o(1)}.$$

It follows that $\binom{r}{t}/\binom{r}{d} \ge x/L(x)^{1+o(1)}$, as desired. ∎

Remark. The kth powers appearing above are all y-smooth. There are only $L(x)^{o(1)}$ y-smooth numbers $n \le x$ (see, e.g., Theorem 5.2 of [13]), so by the pigeonhole principle, some kth power $m \le x$ satisfies $\#\varphi^{-1}(m) \ge x/L(x)^{1+o(1)}$. (Compare with the first half of [2, Theorem 6.2].) Pomerance [10] has shown that $\max_{m\le x} \#\varphi^{-1}(m) \le x/L(x)^{1+o(1)}$. We conclude that there are kth powers that are essentially 'as popular as possible' as φ-values.

3. Preliminaries for the proof of Theorem 2

We quote several results from the author's recent work [8]. Given a real number $z \ge 2$, we let $W_z(\cdot)$ denote the additive arithmetic function whose value at each prime power p^e is given by $W_z(p^e) = \omega_{>z}(\varphi(p^e))$.

Lemma 4 ([8, Lemma 2])**.** *Let*

$$z = \exp((\log_2 x)^{1/2}).$$

Fix any $\eta \in (0,1)$, and let

$$A = (\log_2 x)^{1-\eta}.$$

As $x \to \infty$,

$$\sum_{n\le x} A^{W_z(n)} \le xL(x)^{o(1)}.$$

Lemma 5 ([8, Lemma 3])**.** *Let $Y, Z \ge 1$. The number of positive integers $n \le x$ with*

$$\sum_{p\mid n} (\Omega_{>Y}(p-1) - \omega_{>Y}(p-1)) \ge Z$$

is at most $xL(x)^{2+o(1)}Y^{-Z/2}$, as $x \to \infty$ (uniformly in Y, Z).

Lemma 6 ([8, Lemma 4])**.** *Let $Z \ge 1$. The number of $n \le x$ with*

$$\omega(n) \le \frac{\log x \cdot \log_3 x}{(\log_2 x)^2} \qquad (2)$$

satisfying

$$\sum_{p\mid n} \Omega(p-1) \ge Z$$

is at most

$$xL(x)^{2+o(1)}2^{-Z/2},$$

as $x \to \infty$ (uniformly in Z).

Lemma 7 (see [8, Lemma 5] and the remark following)**.** *The following statement holds for a certain constant $C > 0$: Let $x \ge 3$, and let d be a positive integer. Then the number of positive integers $n \le x$ for which $d \mid \varphi(n)$ is at most*

$$\frac{x}{d}(C(\log_2 x)^2)^{\Omega(d)} \cdot \Omega(d)^{\Omega(d)}.$$

We also require the following exponential moment estimate, which will be used to bound the number of integers possessing many prime factors exceeding $\log x$.

Lemma 8. *Let* $y := \frac{\log x}{(\log_2 x)^2}$. *Then, as* $x \to \infty$,

$$\sum_{n \le T} y^{\Omega_{>\log x}(n)} \le T \cdot x^{o(1)},$$

uniformly for real $T \in [1, x]$.

Proof. Let G be the multiplicative function determined by the convolution identity

$$y^{\Omega_{>\log x}(n)} = \sum_{d|n} G(d).$$

Then G vanishes at prime powers p^a with $p \le \log x$, while if $p > \log x$, then $G(p^a) = y^a - y^{a-1}$. Hence,

$$\begin{aligned}\sum_{n \le T} y^{\Omega_{>\log x}(n)} &= \sum_{d \le T} G(d) \left\lfloor \frac{T}{d} \right\rfloor \\ &\le T \sum_{d \le T} \frac{G(d)}{d} \le T \prod_{\log x < p \le T} \left(1 + \frac{y-1}{p} + \frac{y^2-y}{p^2} + \dots\right).\end{aligned}$$

If $T \le \log x$, the product is empty (and thus equal to 1). So suppose that $T > \log x$. For all $p \in (\log x, T]$, we have that (assuming x is large enough)

$$\frac{y-1}{p} + \frac{y^2-y}{p^2} + \dots = \frac{y-1}{p-y} < \frac{y-1}{p/2} < \frac{2y}{p},$$

and so

$$\begin{aligned}\prod_{\log x < p \le T} \left(1 + \frac{y-1}{p} + \frac{y^2-y}{p^2} + \dots\right) &\le \exp\left(\sum_{\log x < p \le T} \left(\frac{y-1}{p} + \frac{y^2-y}{p^2} + \dots\right)\right) \\ &\le \exp\left(\sum_{\log x < p \le T} \frac{2y}{p}\right) < \exp\left(2y \log_2 T\right) \le \exp(2 \log x / \log_2 x) = x^{o(1)}.\end{aligned}$$

Collecting the estimates completes the proof. □

4. Proof of Theorem 2

In this section, z has the same meaning as in Lemma 4: $z = \exp((\log_2 x)^{1/2})$. We let $\epsilon > 0$ be arbitrary but fixed.

We begin by excluding a set of inconvenient values of n, of cardinality $O(x/L(x)^{1-2\epsilon})$. This part of the argument closely parallels work in [8], but we repeat the details for the convenience of the reader.

We start by assuming that $\omega(n)$ satisfies the inequality (2). By a 1917 theorem of Hardy–Ramanujan [7], the number of $n \le x$ with $\omega(n) > \log x \cdot \log_3 x / (\log_2 x)^2$ is

$$\ll \frac{x}{\log x} \sum_{k > \log x \cdot \log_3 x / (\log_2 x)^2} \frac{(\log_2 x + O(1))^{k-1}}{(k-1)!} \le x/L(x)^{1+o(1)},$$

as $x \to \infty$, which is well within our target upper bound.

Next, we assume that $\varphi(n) > x/L(x)$. Since $\varphi(n) \gg n/\log_2 n$, the number of $n \le x$ with $\varphi(n) \le x/L(x)$ is $O(x \log_2 x / L(x))$, which is acceptable.

We further assume that

$$W_z(n) \le (1-\epsilon)\frac{\log x}{\log_2 x}.$$

By Lemma 4, with $\eta = \epsilon$, this excludes at most $x/L(x)^{(1-\epsilon)^2+o(1)}$ values of x (as $x \to \infty$). Again, this is acceptable.

Next, we assume that

$$\Omega_z(\varphi(n)) \le \left(1 - \frac{1}{2}\epsilon\right)\frac{\log x}{\log_2 x}. \tag{3}$$

If this inequality fails, then

$$\begin{aligned}
\frac{1}{2}\epsilon\frac{\log x}{\log_2 x} &< \Omega_{>z}(\varphi(n)) - W_z(n) \\
&= \sum_{p^e \| n} (\Omega_{>z}(\varphi(p^e)) - \omega_{>z}(\varphi(p^e))) \\
&= \sum_{p^e \| n} (\Omega_{>z}(p^{e-1}) - \omega_{>z}(p^{e-1})) + \sum_{p \mid n} (\Omega_{>z}(p-1) - \omega_{>z}(p-1)) \\
&= \sum_{\substack{p^e \| n \\ e \ge 2 \\ p > z}} (e-2) + \sum_{p \mid n} (\Omega_{>z}(p-1) - \omega_{>z}(p-1)).
\end{aligned}$$

Thus, one of the final two sums exceeds $\frac{1}{4}\epsilon\frac{\log x}{\log_2 x}$. If it is the first sum, then

$$\prod_{\substack{p^e \| n \\ e \ge 2}} p^e \ge \prod_{\substack{p^e \| n \\ e \ge 2 \\ p > z}} z^e > z^{\frac{1}{4}\epsilon\frac{\log x}{\log_2 x}} = \exp\left(\frac{1}{4}\epsilon\frac{\log x}{(\log_2 x)^{1/2}}\right).$$

The number of $n \le x$ with a squarefull divisor of this size does not exceed

$$x \sum_{\substack{m > \exp\left(\frac{1}{4}\epsilon\frac{\log x}{(\log_2 x)^{1/2}}\right) \\ m \text{ squarefull}}} \frac{1}{m} \ll x \exp\left(-\frac{1}{8}\epsilon\frac{\log x}{(\log_2 x)^{1/2}}\right).$$

(The sum on m has been estimated by partial summation, after recalling that the count of squarefull $m \le T$ is $O(T^{1/2})$, for all $T \ge 1$.) In particular, the number of these n is smaller than $x/L(x)$ for large x. If the second of the two sums exceeds $\frac{1}{4}\epsilon\frac{\log x}{\log_2 x}$, then Lemma 5 (with $Y = z$ and $Z = \frac{1}{4}\epsilon\frac{\log x}{\log_2 x}$) shows that n belongs to a set of size at most

$$xL(x)^{2+o(1)}Y^{-Z/2} = xL(x)^{2+o(1)}\exp\left(-\frac{1}{8}\epsilon\frac{\log x}{(\log_2 x)^{1/2}}\right),$$

which is again smaller than $x/L(x)$ for large x.

Finally, we assume that

$$\Omega(\varphi(n)) \le \frac{\log x}{(\log_2 x)^{2/3}}. \tag{4}$$

Supposing this inequality fails, we have that

$$\sum_{p \mid n} \Omega(p-1) + \sum_{p^e \| n,\ e \ge 2} (e-1) = \sum_{p \mid n} \Omega(\varphi(p^e)) = \Omega(\varphi(n)) > \frac{\log x}{(\log_2 x)^{2/3}}.$$

So one of the two sums on the far left exceeds $\frac{1}{2}\frac{\log x}{(\log_2 x)^{2/3}}$. If it is the latter, then n is divisible by a squarefull number exceeding $2^{\frac{1}{2}\log x/(\log_2 x)^{2/3}}$. The number of such $n \le x$ does not exceed

$$x \sum_{\substack{m > 2^{\frac{1}{2}\log x/(\log_2 x)^{2/3}} \\ m \text{ squarefull}}} \frac{1}{m} \ll x \exp\left(-\frac{\log 2}{4}\frac{\log x}{(\log_2 x)^{2/3}}\right);$$

this is smaller than $x/L(x)$ for large x. If it is the former, we apply Lemma 6 to see that n is placed in a certain set of size at most

$$xL(x)^{2+o(1)} \exp\left(-\frac{\log 2}{4}\frac{\log x}{(\log_2 x)^{2/3}}\right),$$

which is again smaller than $x/L(x)$ for large x.

We will show that the number of $n \le x$ for which $\varphi(n)$ is squarefull, and for which all of the above assumptions are satisfied, is smaller than $x^{1-\frac{1}{10}\epsilon}$. As this is $o(x/L(x))$, we conclude that the total number of $n \le x$ with $\varphi(n)$ squarefull is $O(x/L(x)^{1-2\epsilon})$. Since ϵ may be taken arbitrarily small, the theorem follows.

Every squarefull integer can be decomposed as a^2b^3, with a, b positive integers. Hence, we may write

$$\varphi(n) = a_1^2 a_2^2 a_3^2 b_1^3 b_2^3 b_3^3$$

where

$$\begin{aligned} p \mid a_1 b_1 &\Rightarrow p \le z, \\ p \mid a_2 b_2 &\Rightarrow z < p \le \log x, \\ p \mid a_3 b_3 &\Rightarrow \log x < p. \end{aligned}$$

From the definition of z together with (4),

$$a_1^2 b_1^3 \le z^{\Omega(a_1^2 b_1^3)} \le z^{\Omega(\varphi(n))} \le \exp\left(\frac{\log x}{(\log_2 x)^{1/6}}\right) = x^{o(1)}.$$

Since $\varphi(n) > x/L(x)$, we deduce that

$$a_2^2 a_3^2 b_2^3 b_3^3 > \frac{x}{L(x) a_1^2 b_1^3} = x^{1+o(1)}. \tag{5}$$

Put $R = \lceil \log x \rceil$ and $\delta = 1/R$. Choose $\alpha_2, \alpha_3, \beta_2, \beta_3 \in \{1/R, 2/R, \ldots, (R-1)/R, 1\}$ such that, for $i = 2, 3$,

$$a_i \in [x^{\alpha_i - \delta}, x^{\alpha_i}], \quad \text{while} \quad b_i \in [x^{\beta_i - \delta}, x^{\beta_i}].$$

Note that $x^{1/R} \ll 1$, and so $a_i \asymp x^{\alpha_i}, b_i \asymp x^{\beta_i}$. By (5),

$$2\alpha_2 + 2\alpha_3 + 3\beta_2 + 3\beta_3 \ge 1 + o(1). \tag{6}$$

Also, recalling (3),

$$2\Omega(a_2) + 2\Omega(a_3) + 3\Omega(b_2) + 3\Omega(b_3) = \Omega_z(\varphi(n)) \le \left(1 - \frac{1}{2}\epsilon\right)\frac{\log x}{\log_2 x}. \tag{7}$$

Switching perspective, we may view $a_1, a_2, a_3, b_1, b_2, b_3$ as given, and count the number of corresponding values of $n \le x$. By Lemma 7,

$$\#\{n \le x : \varphi(n) = a_1^2 a_2^2 a_3^2 b_1^3 b_2^3 b_3^3\} \le \#\{n \le x : a_3^2 b_3^3 \mid \varphi(n)\}$$
$$\le \frac{x}{a_3^2 b_3^3} (C(\log_2 x)^2)^{\Omega(a_3^2 b_3^3)} \Omega(a_3^2 b_3^3)^{\Omega(a_3^2 b_3^3)}.$$

Keeping (7) in mind, and using that $a_i \asymp x^{\alpha_i}, b_i \asymp x^{\beta_i}$, this upper bound is seen to be at most

$$x^{1-2\alpha_3-3\beta_3+o(1)} (\log x)^{\Omega(a_3^2 b_3^3)} = x^{1-2\alpha_3-3\beta_3+o(1)} (\log x)^{2A_3+3B_3},$$

where we define $A_i = \Omega(a_i)$ and $B_i = \Omega(b_i)$.

Our strategy for the rest of the proof is as follows: We fix values for α_3, β_3, A_3, and B_3. This puts an additional constraint on the tuples $a_1, a_2, a_3, b_1, b_2, b_3$; namely, it amounts to specifying the order of magnitude and the number of prime factors of a_3 and b_3. In the next paragraph, we bound from above the number of tuples $a_1, a_2, a_3, b_1, b_2, b_3$ obeying this constraint. Each such tuple corresponds to at most $x^{1-2\alpha_3-3\beta_3+o(1)} (\log x)^{2A_3+B_3}$ values of n. Thus, multiplying our bound on the number of these tuples by $x^{1-2\alpha_3-3\beta_3+o(1)} (\log x)^{2A_3+B_3}$ yields a bound on the number of n corresponding to this particular choice of α_3, β_3, A_3, and B_3. (As we will see, some additional effort is required to suss out the size of this bound.) Finally, we bound the total number of n by summing on α_3, β_3, A_3, and B_3. We emphasize that all $o(1)$ terms appearing here and below are to be understood with $x \to \infty$, and that the convergence of $o(1)$ to 0 is uniform in all other parameters.

Let y be as in Lemma 8, i.e., $y = \frac{\log x}{(\log_2 x)^2}$. The total number of $(\log x)$-smooth integers in $[1, x]$ is $x^{o(1)}$ (see again Theorem 5.2 of [13]), and so there are at most $x^{o(1)}$ possibilities for each of a_1, b_1, a_2, b_2. We have that $a_3 \le x^{\alpha_3}$ (with α_3 fixed), that every prime factor of a_3 exceeds $\log x$, and that $\Omega(a_3)$ takes the fixed value A_3; thus, the number of possibilities for a_3 is at most

$$y^{-A_3} \sum_{m \le x^{\alpha_3}} y^{\Omega_{>\log x}(m)} \le y^{-A_3} x^{\alpha_3+o(1)}.$$

Similarly, the number of possibilities for b_3 is at most $y^{-B_3} x^{\beta_3+o(1)}$. Thus, the number of n corresponding to a fixed choice of $\alpha_3, \beta_3, A_3, B_3$ is at most

$$x^{1-2\alpha_3-3\beta_3+o(1)} (\log x)^{2A_3+3B_3} \cdot x^{\alpha_3+\beta_3+o(1)} y^{-A_3-B_3}. \tag{8}$$

Assume that there is some tuple $a_1, a_2, a_3, b_1, b_2, b_3$ corresponding to our fixed choice of α_3, β_3, A_3, and B_3. (If this fails, then there are no corresponding n.) Choose one and define, for $i = 2, 3$,

$$\tilde{\alpha}_i = \frac{A_i}{\log x / \log_2 x}, \quad \text{and} \quad \tilde{\beta}_i = \frac{B_i}{\log x / \log_2 x}.$$

(Since A_3, B_3 are fixed, the values of $\tilde{\alpha}_3$ and $\tilde{\beta}_3$ do not depend on the choice of tuple $a_1, a_2, a_3, b_1, b_2, b_3$, but the values of $\tilde{\alpha}_2$ and $\tilde{\beta}_2$ may.) By (7),

$$2\tilde{\alpha}_2 + 2\tilde{\alpha}_3 + 3\tilde{\beta}_2 + 3\tilde{\beta}_3 \le 1 - \frac{1}{2}\epsilon. \tag{9}$$

For both $i = 2, 3$, we have that $(\log x)^{A_i} = x^{\tilde{\alpha}_i}$, $(\log x)^{B_i} = x^{\tilde{\beta}_i}$, $y^{A_i} = x^{\tilde{\alpha}_i+o(1)}$, and $y^{B_i} = x^{\tilde{\beta}_i+o(1)}$ (as $x \to \infty$); making these substitutions above transforms our upper bound (8) into

$$x^{1-(\alpha_3-\tilde{\alpha}_3)-2(\beta_3-\tilde{\beta}_3)+o(1)}. \tag{10}$$

How large is the exponent here? Subtracting (9) from (6),

$$2(\alpha_2-\tilde{\alpha}_2)+2(\alpha_3-\tilde{\alpha}_3)+3(\beta_2-\tilde{\beta}_2)+3(\beta_3-\tilde{\beta}_3)\geq \frac{1}{2}\epsilon+o(1);$$

hence,

$$2(\alpha_3-\tilde{\alpha}_3)+3(\beta_3-\tilde{\beta}_3)\geq \frac{1}{2}\epsilon+2(\tilde{\alpha}_2-\alpha_2)+3(\tilde{\beta}_2-\beta_2)+o(1).$$

Since every prime factor of a_2b_2 is at most $\log x$, we see that $x^{\alpha_2-\delta}\leq a_2\leq(\log x)^{A_2}=x^{\tilde{\alpha}_2}$; thus, $\alpha_2\leq\tilde{\alpha}_2+o(1)$. Similarly, $\beta_2\leq\tilde{\beta}_2+o(1)$. Therefore,

$$2(\alpha_3-\tilde{\alpha}_3)+3(\beta_3-\tilde{\beta}_3)\geq \frac{1}{2}\epsilon+o(1).$$

Since each prime factor of a_3, b_3 exceeds $\log x$, we find that $\alpha_3\geq\tilde{\alpha}_3$ and $\beta_3\geq\tilde{\beta}_3$; so both of the left-hand summands in the last display are nonnegative. Thus, either $\alpha_3-\tilde{\alpha}_3\geq\frac{1}{8}\epsilon+o(1)$ or $\beta_3-\tilde{\beta}_3\geq\frac{1}{12}\epsilon+o(1)$. Putting this back into (10), we find that the number of n corresponding to a fixed choice of α_3,β_3,A_3,B_3 is at most

$$x^{1-\frac{1}{8}\epsilon+o(1)}.$$

It remains to sum on α_3,β_3,A_3,B_3. There are (crudely) only $O(\log x)$ possibilities for each of these parameters. Hence, the total number of n satisfying our assumptions for which $\varphi(n)$ squarefull is at most $x^{1-\frac{1}{8}\epsilon+o(1)}(\log x)^4$, and so is smaller than $x^{1-\frac{1}{10}\epsilon}$ for large x, as desired.

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