POWERFREE SUMS OF PROPER DIVISORS

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Abstract. Let $s(n) := \sum_{d|n,\ d<n} d$ denote the sum of the proper divisors of n. It is natural to conjecture that for each integer $k \geq 2$, the equivalence

$$n \text{ is } k\text{th powerfree} \Longleftrightarrow s(n) \text{ is } k\text{th powerfree}$$

holds almost always (meaning, on a set of asymptotic density 1). We prove this for $k \geq 4$.

1. Introduction

A 19th century theorem of Gegenbauer asserts that for each fixed k, the set of positive integers not divisible by the kth power of an integer larger than 1 has asymptotic density $\zeta(k)^{-1}$, where $\zeta(s)$ is the familiar Riemann zeta function. Recall that the asymptotic density of a set $\mathcal{A}$ of positive integers is the limiting proportion of the elements of $\mathcal{A}$ up to x, more precisely the limit as $x \to \infty$ of the quantity $\frac{1}{x}\#\{a \leq x : a \in \mathcal{A}\}$, subject to existence.

Call an integer kth-power-free, or k-free for short when it is not divisible by the kth power of an integer larger than 1. In this note we investigate the frequency with which the sum-of-proper-divisors function $s(n) := \sum_{d|n,\ d<n} d$ assumes k-free values. As we proceed to explain, there is a natural guess to make here, formulated below as Conjecture 1.1.

Fix $k \geq 2$. If n is not k-free, then $p^k \mid n$ for some prime p. Moreover, if $y = y(x)$ is any function tending to infinity, the the upper density of n divisible by p^k for some $p > y^{1/k}$ is at most $\sum_{p>y^{1/k}} p^{-k} = o(1)$. Hence, almost always a non k-free number n is divisible by p^k for some $p^k \leq y$. To be precise, when we say a statement about positive integers n holds almost always, we mean that it holds for all $n \leq x$ with $o(x)$ exceptions, as $x \to \infty$. (Importantly, we allow the statement itself to involve the growing upper bound x.)

It was noticed by Alaoglu and Erdős [AE44] that whenever $y = y(x)$ tends to infinity with x slowly enough, $\sigma(n)$ is divisible by all of the integers in $[1, y]$ almost always. (We give a proof below with $y := (\log\log x)^{1-\epsilon}$; see Lemma 2.2.) Hence, almost always n and $s(n) = \sigma(n) - n$ share the same set of divisors up to y. Putting this together with the observations of the last paragraph, we see that if n is not k-free, then $s(n)$ is not k-free, almost always. The same reasoning shows that if n is k-free, then $s(n)$ is not divisible by p^k for any $p \leq y^{1/k}$, almost always. Thus, if it could be shown that almost always $s(n)$ is not divisible by p^k for any prime $p > y^{1/k}$, then we would have established the following conjecture.

Conjecture 1.1. *Fix $k \geq 2$. On a set of integers n of asymptotic density 1,*

$$n \text{ is } k\text{-free} \Longleftrightarrow s(n) \text{ is } k\text{-free}.$$

2020 *Mathematics Subject Classification.* Primary 11N37; Secondary 11A25, 11N64.

Key words and phrases. squarefree, powerfree, k-free, sum of proper divisors, sum of aliquot divisors.

The case $k = 2$ of Conjecture 1.1 is alluded to by Luca and Pomerance in [LP15] (see Lemma 2.2 there and the discussion following). Their arguments show that $s(n)$ is squarefree on a set of positive lower density (in fact, of lower density at least $\zeta(2)^{-1} \log 2$). Conjecture 1.1, for every $k \geq 2$, would then be a consequence of the following general conjecture of Erdős–Granville–Pomerance–Spiro [EGPS90] (see Remark 3.5 below).

Conjecture 1.2. *If $\mathcal{A}$ is a set of natural numbers of positive upper density, then $s(\mathcal{A}) := \{s(n) : n \in \mathcal{A}\}$ also has positive upper density.*

We recall that the upper and lower densities of a set of positive integers are defined in the exact same way as the asymptotic density, but with $\limsup$ and $\liminf$ replacing the limit respectively (so that these always exist).

Our result is as follows.

Theorem 1.3. *Conjecture 1.1 holds for each $k \geq 4$.*

To prove Conjecture 1.1 for a given k, it is enough (by the above discussion) to show that almost always $s(n)$ is not divisible by p^k for any $p^k > (\log\log x)^{0.9}$. The range $p \leq x^{o(1)}$ can be treated quickly using familiar arguments (versions of which appear, e.g., in [Pol14]). The main innovation in our argument — and the source of the restriction to $k \geq 4$ — is the handling of larger p using a theorem of Wirsing [Wir59] that bounds the "popularity" of values of the function $\sigma(n)/n$.

The reader interested in other work on powerfree values of arithmetic functions may consult [Pap03, PSS03, BL05, BP06] as well as the survey [Pap05].

Notation and conventions. We reserve the letters p, q, P, with or without subscripts, for primes and we write $\log_k$ for the kth iterate of the natural logarithm. We write $P^+(n)$ and $P^-(n)$ for the largest and smallest prime factors of n, with the conventions that $P^+(1) = 1$ and $P^-(1) = \infty$. We adopt the Landau–Bachmann–Vinogradov notation from asymptotic analysis, with all implied constants being absolute unless specified otherwise.

2. Preliminaries

The following lemma is due to Pomerance (see [Pom77, Theorem 2]).

Lemma 2.1. *Let a, k be integers with $\gcd(a, k) = 1$ and $k > 0$. Let $x \geq 3$. The number of $n \leq x$ for which there does not exist a prime $p \equiv a \pmod{k}$ for which $p \parallel n$ is $O(x(\log x)^{-1/\varphi(k)})$.*

The next lemma justifies the claim in the introduction that $\sigma(n)$ is usually divisible by all small primes. It is well-known (see, e.g., Lemma 2.1 of [LP15]) but we include the short proof.

Lemma 2.2. *Fix $\epsilon > 0$. Almost always, the number $\sigma(n)$ is divisible by every positive integer $d \leq (\log_2 x)^{1-\epsilon}$.*

Proof. Notice that $d \mid \sigma(n)$ whenever there is a prime $p \equiv -1 \pmod{d}$ such that $p \parallel n$. For each $d \leq (\log_2 x)^{1-\epsilon}$, the number of $n \leq x$ for which there is no such p is $O(x \exp(-(\log_2 x)^{\epsilon}))$, by Lemma 2.1. Now sum on $d \leq (\log_2 x)^{1-\epsilon}$. ☐

Our next lemma bounds the number of $n \le x$ for which n and $\sigma(n)$ possess a large common prime divisor. In what follows, we say that a positive integer a is squarefull if no prime appears only to the first power in a; or, in other words, if p^2 divides a for every prime p dividing a. By the squarefull part of a natural number, we shall mean its largest squarefull divisor.

Lemma 2.3. *Almost always, the greatest common divisor of n and $\sigma(n)$ has no prime divisor exceeding $\log_2 x$.*

With more effort, it could be shown that $\gcd(n, \sigma(n))$ is almost always the largest divisor of n supported on primes not exceeding $\log_2 x$. Compare with Theorem 8 in [ELP08], which is the corresponding assertion with $\sigma(n)$ replaced by $\varphi(n)$.

Proof. Put $y := \log_2 x$. We start by removing those $n \le x$ with squarefull part exceeding $\frac{1}{2}y$. The number of these n is $O(xy^{-1/2})$, which is $o(x)$ and hence negligible.

Suppose that n survives and there is a prime $p > y$ dividing n and $\sigma(n)$. Since $p \mid \sigma(n)$, we can choose a prime power $q^e \parallel n$ for which $p \mid \sigma(q^e)$. Then $y < p \le \sigma(q^e) < 2q^e$, forcing $e = 1$. Hence, $p \mid \sigma(q) = q + 1$ and $q \equiv -1 \pmod p$. Since $pq \mid n$, we deduce that the number of n belonging to this case is at most

$$\sum_{p>y} \sum_{\substack{q \equiv -1 \ (\mathrm{mod}\ p) \\ q \le x}} \frac{x}{pq} \ll x \sum_{p>y} \frac{1}{p} \sum_{\substack{q \le x \\ q \equiv -1 \ (\mathrm{mod}\ p)}} \frac{1}{q} \ll x \log_2 x \sum_{p>y} \frac{1}{p^2} \ll \frac{x \log_2 x}{y \log y} = \frac{x}{\log_3 x},$$

which is again $o(x)$. Here the sum on q has been estimated by the Brun–Titchmarsh inequality (see, e.g., Theorem 416 on p. 83 of [Ten15]) and partial summation. $\square$

The next lemma bounds the number of $n \le x$ with two large prime factors that are multiplicatively close.

Lemma 2.4. *For all large x, the number of $n \le x$ divisible by a pair of primes q_1, q_2 with*

$$x^{1/10 \log_3 x} < q_1 \le x \quad \text{and} \quad q_1 x^{-1/(\log_3 x)^2} \le q_2 \le q_1$$

is $O(x/\log_3 x)$.

Proof. The number of such n is at most $x \sum_{x^{1/10 \log_3 x} < q_1 \le x} \frac{1}{q_1} \sum_{q_1 x^{-1/(\log_3 x)^2} \le q_2 \le q_1} \frac{1}{q_2}$. By Mertens' theorem, the inner sum is

$$\ll \log\left(\frac{\log(q_1)}{\log(q_1 x^{-1/(\log_3 x)^2})}\right) + \frac{1}{\log(q_1 x^{-1/(\log_3 x)^2})} \ll \frac{\log x}{(\log q_1)(\log_3 x)^2},$$

leading to an upper bound for our count of n of

$$\ll \frac{x \log x}{(\log_3 x)^2} \sum_{x^{1/10 \log_3 x} < q_1 \le x} \frac{1}{q_1 \log q_1} \ll \frac{x \log x}{(\log_3 x)^2} \cdot \frac{\log_3 x}{\log x} = \frac{x}{\log_3 x}.$$

Here the final sum has been estimated by the prime number theorem and partial summation. $\square$

We conclude this section by quoting the main result of [Wir59].

Lemma 2.5 (Wirsing). *There exists an absolute constant $\lambda_0 > 0$ such that*
$$\#\left\{m \le x : \frac{\sigma(m)}{m} = \alpha\right\} \le \exp\left(\lambda_0 \frac{\log x}{\log_2 x}\right)$$
for all $x \ge 3$ and all real numbers α.

3. Proof of Theorem 1.3

As discussed in the introduction, it is enough to establish the following proposition. From now on, $y := (\log_2 x)^{0.9}$.

Proposition 3.1. *Fix $k \ge 4$. Almost always, $s(n)$ is not divisible by p^k for any $p^k > y$.*

We split the proof of Proposition 3.1 into two parts, according to the size of p.

3.1. **. . . when $y < p^k \le x^{1/2\log_3 x}$.** The following is a weakened form of Lemma 2.8 from [Pol14].

Lemma 3.2. *For all large x, there is a set $\mathcal{E}(x)$ having size $o(x)$, as $x \to \infty$, such that the following holds. For all $d \le x^{1/2\log_3 x}$, the number of $n \le x$ not belonging to $\mathcal{E}(x)$ for which $d \mid s(n)$ is $O(x/d^{0.9})$.*

Summing the bound of Lemma 3.2 over $d = p^k$ with $y < p^k \le x^{1/2\log_3 x}$ gives $o(x)$. It follows that almost always, $s(n)$ is not divisible by p^k for any $p^k \in (y, x^{1/2\log_3 x}]$.

3.2. **. . . when $p^k > x^{1/2\log_3 x}$.** The treatment of this range of p is based on the following result, which may be of independent interest.

Theorem 3.3. *For all large x, there is a set $\mathcal{E}(x)$ having size $o(x)$, as $x \to \infty$, such that the following holds. The number of $n \le x$ not belonging to $\mathcal{E}(x)$ for which $d \mid s(n)$ is*
$$\ll \frac{x}{d^{1/4}\log x}$$
uniformly for positive integers $d > x^{1/2\log_3 x}$ satisfying $P^-(d) > \log_2 x$.

The crucial advantage of Theorem 3.3 over Lemma 3.2 is the lack of any restriction on the size of d. Since $k \ge 4$, when we sum the bound of Theorem 3.3 over $d = p^k$ with $x^{1/2\log_3 x} < p^k < x^2$, the result is $O(x\log_2 x/\log x)$, which is $o(x)$. So the proof of Theorem 1.3 will be completed once Theorem 3.3 is established.

Turning to the proof of Theorem 3.3, let $\mathcal{E}(x)$ denote the collection of $n \le x$ for which at least one of the following fails:

(1) $n > x/\log x$,

(2) the largest squarefull divisor of n is at most $\log_2 x$,

(3) $P^+(n) > x^{1/10\log_3 x}$,

(4) $P^+(n)^2 \nmid n$,

(5) $P^+(\gcd(n, \sigma(n))) \le \log_2 x$,

(6) $P^+(n) > P_2^+(n)x^{1/(\log_3 x)^2}$, where $P_2^+(n) := P^+(n/P^+(n))$ is the second-largest prime factor of n.

Let us show that only $o(x)$ integers $n \le x$ fail one of (1)–(6). This is obvious for (1). The count of $n \le x$ failing (2) is $\ll x \sum_{r > \log_2 x,\ r \text{ squarefull}} 1/r \ll x/\sqrt{\log_2 x}$, and thus is $o(x)$. That the count of $n \le x$ failing (3) is $o(x)$ follows from standard bounds on the counting function of smooth (friable) numbers (e.g., Theorem 5.1 on p. 512 of [Ten15]), or Brun's sieve. The set of $n \le x$ passing (3) but failing (4) has cardinality $\ll x \sum_{r > x^{1/10 \log_3 x}} 1/r^2 = o(x)$. Condition (5) is handled by Lemma 2.3. That the count of $n \le x$ satisfying (1)–(5) and failing (6) is $o(x)$ follows from Lemma 2.4.

Let d be as in Theorem 3.3. We separate the count of $n \notin \mathcal{E}(x)$ for which $d \mid s(n)$ according to whether $P^+(n) < d^{1/4}(\log x)^2$ or $P^+(n) \ge d^{1/4}(\log x)^2$.

We first consider $n \notin \mathcal{E}(x)$ with $P^+(n) \ge d^{1/4}(\log x)^2$. Write $n = mP$, where $P := P^+(n)$. Then $\gcd(m, P) = 1$, and

$$x/m \ge d^{1/4}(\log x)^2.$$

We can rewrite the condition $d \mid s(n)$ as

$$Ps(m) \equiv -\sigma(m) \pmod{d}.$$

For this congruence to have solutions, we must have $\gcd(s(m)\sigma(m), d) = 1$. Indeed, if there exists a prime q dividing both $\sigma(m)$ and d, then from $q \mid d$, we have $q > \log_2 x$, whereas since $d \mid s(n)$, we also have $q \mid s(n)$. But then the divisibility $q \mid \sigma(m) \mid \sigma(n)$ leads to $q \mid \gcd(n, \sigma(n))$, contradicting condition (5) above. Since any common prime divisor of $s(m)$ and d would, by the congruence, have to divide $\sigma(m)$ as well, we must indeed have $\gcd(s(m)\sigma(m), d) = 1$.

As such, the above congruence condition on P places it in a unique coprime residue class modulo d. Hence, given m, the number of possible P (and hence possible $n = mP$) is

$$\ll \frac{x}{md} + 1 \ll \frac{x}{md} + \frac{x}{md^{1/4}(\log x)^2},$$

which when summed over $m \le x$ is $\ll x/d^{1/4}\log x$, consistent with Theorem 3.3. (We use here the lower bound on d.)

It remains to count $n \le x$, $n \notin \mathcal{E}(x)$ where $d \mid s(n)$ and $P^+(n) < d^{1/4}(\log x)^2$. For this case, we fix a constant

$$\lambda > 2\lambda_0,$$

where λ_0 is the constant appearing in Wirsing's bound (Lemma 2.5). We will assume that $d \le x^{3/2}$, since $s(n) \le \sigma(n) < x^{3/2}$ for all $n \le x$, once x is sufficiently large (e.g., as a consequence of the bound $\sigma(n) \ll n \log_2(3n)$; see Theorem 323 in [HW08]).

We write $n = AB$, where A is the least unitary squarefree divisor of $n/P^+(n)$ exceeding $d^{1/4}\exp\left(\frac{\lambda}{2}\frac{\log x}{\log_2 x}\right)$. Such a divisor exists as $n > x/\log x$ has maximal squarefull divisor at most $\log_2 x$, whereupon its largest unitary squarefree divisor coprime to $P^+(n)$ must be no less than

$$\frac{1}{d^{1/4}(\log x)^2 \log_2 x} \cdot \frac{x}{\log x} > d^{1/4}\exp\left(\frac{\lambda}{2}\frac{\log x}{\log_2 x}\right).$$

(We assume throughout this argument that x is sufficiently large.) Then

$$B \le \frac{x}{A} \le \frac{x}{d^{1/4}} \exp\left(-\frac{\lambda}{2}\frac{\log x}{\log_2 x}\right). \tag{1}$$

Furthermore,

$$P^+(A) \le P_2^+(n) < P^+(n)x^{-1/(\log_3 x)^2} < d^{1/4}(\log x)^2 x^{-1/(\log_3 x)^2} < d^{1/4}x^{-\lambda/\log_2 x}.$$

Since $A/P^+(A)$ is a unitary squarefree divisor of $n/P^+(n)$, to avoid contradicting the choice of A, we must have $A \le d^{1/2}\exp\left(-\frac{\lambda}{2}\frac{\log x}{\log_2 x}\right)$. Then $\sigma(A) \ll A\log_2 A \ll A\log_2 x$, so that (for large x) $\sigma(A) < d^{1/2}$.

For each B as above, we bound the number of corresponding A. First of all, since $\gcd(A,B)=1$, the divisibility $d \mid s(n)$ translates to the congruence $\sigma(A)\sigma(B) \equiv AB \pmod d$. Now, we claim that $\gcd(A\sigma(B), d) = 1$: indeed, for any prime q dividing both A and d, we must have, on one hand, $q \ge P^-(d) > \log_2 x$, while on the other, $q \mid d \mid s(n)$ and $q \mid A \mid n$ imply $q \mid \gcd(n, \sigma(n))$. This contradicts (5). It follows by an analogous argument that $\gcd(\sigma(B), d) = 1$, thus proving our claim. Consequently, the above congruence may be rewritten as

$$\frac{\sigma(A)}{A} \equiv \frac{B}{\sigma(B)} \pmod d.$$

Now for some B, consider any pair of squarefree integers A_1 and A_2 satisfying the above congruence along with the conditions $\sigma(A_1), \sigma(A_2) < d^{1/2}$. Then $\sigma(A_1)/A_1 \equiv \sigma(A_2)/A_2 \pmod d$, leading to $\sigma(A_1)A_2 \equiv A_1\sigma(A_2) \pmod d$. But also

$$|\sigma(A_1)A_2 - A_1\sigma(A_2)| \le \max\{\sigma(A_1)A_2, A_1\sigma(A_2)\} < d,$$

thereby forcing $\sigma(A_1)/A_1 = \sigma(A_2)/A_2$. This shows that for each B, all corresponding A have $\sigma(A)/A$ assume the same value, whereupon Lemma 2.5 bounds the number of possible A by $\exp\left(\lambda_0\frac{\log x}{\log_2 x}\right)$. Keeping in mind the upper bound (1) on B, we deduce that the number of n falling into this case is at most

$$\frac{x}{d^{1/4}}\exp\left(-\frac{\lambda}{2}\frac{\log x}{\log_2 x}\right)\cdot\exp\left(\lambda_0\frac{\log x}{\log_2 x}\right) = \frac{x}{d^{1/4}}\exp\left(\left(\lambda_0 - \frac{\lambda}{2}\right)\frac{\log x}{\log_2 x}\right).$$

Since $\lambda > 2\lambda_0$, this final quantity is $\ll x/d^{1/4}\log x$. This completes the proof of Theorem 3.3, and so also that of Theorem 1.3.

Remark 3.4. It is to be noted that one needs the condition $k \ge 4$ in order to have the sum

$$\sum_{x^{1/2\log_3 x} < p^k < x^2} \frac{x}{p^{k/4}\log x},$$

which arises from summing our upper bound in Theorem 3.3 over all $d := p^k > x^{1/2\log_3 x}$, be $o(x)$. Indeed, if $k \le 3$, then this sum would be $\gg x^{7/6}/(\log x)^2$.

Remark 3.5. The conjecture of Erdős, Granville, Pomerance, and Spiro (quoted above as Conjecture 1.2) can be restated as saying that $s^{-1}(\mathcal{A})$ has density 0 whenever $\mathcal{A}$ has density 0. If this holds, then the conclusion of Proposition 3.1 follows for each $k \ge 2$: take

$$\mathcal{A} = \{n \text{ divisible by } p^k \text{ for some } p^k > \log_3(100n)\}.$$

Unfortunately, very little is known in the direction of the EGPS conjecture. The record result (still quite weak) seems to be that of [PPT18], where it is shown that $s^{-1}(\mathcal{A})$ has density 0 whenever $\mathcal{A}$ has counting function bounded by $x^{1/2+o(1)}$, as $x \to \infty$.

Acknowledgements

We thank the referee for helpful comments that improved the exposition. P.P. is supported by NSF award DMS-2001581.

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