PRIME-PERFECT NUMBERS

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In memory of John Lewis Selfridge

Abstract

We discuss a relative of the perfect numbers for which it is possible to prove that there are infinitely many examples. Call a natural number n *prime-perfect* if n and $\sigma(n)$ share the same set of distinct prime divisors. For example, all even perfect numbers are prime-perfect. We show that the count $N_\sigma(x)$ of prime-perfect numbers in $[1, x]$ satisfies estimates of the form

$$\exp((\log x)^{c/\log\log\log x}) \le N_\sigma(x) \le x^{\frac{1}{3}+o(1)},$$

as $x \to \infty$. We also discuss the analogous problem for the Euler function. Letting $N_\varphi(x)$ denote the number of $n \le x$ for which n and $\varphi(n)$ share the same set of prime factors, we show that as $x \to \infty$,

$$x^{7/20} \le N_\varphi(x) \le \frac{x^{1/2}}{L(x)^{1/4+o(1)}}, \quad \text{where} \quad L(x) = x^{\log\log\log x/\log\log x}.$$

We conclude by discussing some related problems posed by Harborth and Cohen.

1. Introduction

Let $\sigma(n) := \sum_{d|n} d$ be the sum of the proper divisors of n. A natural number n is called *perfect* if $\sigma(n) = 2n$ and, more generally, *multiply perfect* if $n \mid \sigma(n)$. The study of such numbers has an ancient pedigree (surveyed, e.g., in [5, Chapter 1] and [28, Chapter 1]), but many of the most interesting problems remain unsolved. Chief among them is the question of whether or not there are infinitely many multiply perfect numbers.

In this note we introduce a class of numbers whose definition is inspired by the perfect numbers but for which we can prove that there are infinitely many examples. Call n *prime-perfect* if $\sigma(n)$ and n have the same set of distinct prime factors. Every even perfect number n is prime-perfect, but there are many other examples, the first being the multiply-perfect number $n = 120$. Prime-perfect numbers appear to have been first considered by the second author, who proved [24] that every such number with two distinct prime factors is an even perfect number.

Our central objective is to establish both lower and upper bounds for $N_\sigma(x)$, the number of prime-perfect $n \le x$. We begin with the lower bound. In what follows, we write $\log_1 x = \max\{1, \log x\}$, and we let $\log_k$ denote the kth iterate of $\log_1$.

Theorem 1.1. *As $x \to \infty$,*

$$N_\sigma(x) \ge \exp((\log x)^{(\frac{1}{2}\log 2 + o(1))/\log_3 x}).$$

We note that this lower bound, though of the shape $x^{o(1)}$, exceeds any fixed power of $\log x$ for x sufficiently large.

Our upper-bound proof covers a class of numbers somewhat wider than that of the prime-perfects. Let $\mathrm{rad}(n)$ denote the largest squarefree divisor of n, so that n is prime-perfect if and only if $\mathrm{rad}(n) = \mathrm{rad}(\sigma(n))$. Call n *prime-abundant* if every prime dividing n divides $\sigma(n)$, i.e., if $\mathrm{rad}(n) \mid \mathrm{rad}(\sigma(n))$. For example, if $n = 2^3 \cdot 3$, then $\sigma(n) = 2^2 \cdot 3 \cdot 5$, so n is prime-abundant but not prime-perfect.

Theorem 1.2. *The number of prime-abundant $n \le x$ is at most $x^{1/3+o(1)}$, as $x \to \infty$.*

The second author conjectured (ca. 1973, unpublished) that a much stronger upper bound should hold for prime-perfect numbers:

Conjecture 1.3. *For each $\epsilon > 0$, we have $N_\sigma(x) = o(x^\epsilon)$, as $x \to \infty$.*

For some numerical perspective, up to 10^9, there are 198 prime-perfect numbers and 5328 prime-abundant numbers.

For perfect and multiply perfect numbers, the analogues of Conjecture 1.3 are known; these are due to Hornfeck and Wirsing [16] (see also [29], whose main result is quoted as Theorem C below). It seems that the prime-perfect setting is genuinely more difficult. One hint as to why is discussed in §4.

Call the natural number n *φ-perfect* if n and $\varphi(n)$ share the same set of prime factors, and let $N_\varphi(x)$ be the corresponding counting function. While analogues of the the Hornfeck–Wirsing results are easily proved if σ is replaced by Euler's φ-function, we show in §4 that $N_\varphi(x)$ does *not* satisfy the bound of Conjecture 1.3. In fact, we have the following estimates:

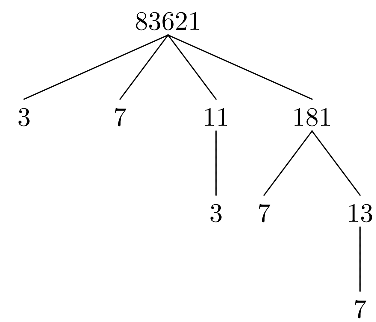

Figure 1: A picture of $\mathscr{T}^{(+)}(83621)$.

Theorem 1.4. *As $x \to \infty$,*

$$x^{7/20} \leq N_\varphi(x) \leq \frac{x^{1/2}}{L(x)^{1/4+o(1)}}, \quad \textit{where} \quad L(x) := x^{\log_3 x/\log_2 x}.$$

In §5, we adapt our methods to study certain problems of Harborth and Cohen. Some questions related to ours are also considered in the papers [18], [19] of Luca.

Notation

Throughout, p and q always denote prime numbers. We let $d(n) := \sum_{d|n} 1$ denote the number of positive divisors of n, while $\omega(n) := \sum_{p|n} 1$ denotes the corresponding count of distinct prime divisors. Let $P(n)$ denote the largest prime divisor of n, with the understanding that $P(1) = 1$. We say that n is *y-smooth* if $P(n) \leq y$, and we write $\Psi(x, y)$ for the count of $n \leq x$ with $P(n) \leq y$. For each n, its *y-smooth part* is defined as the largest y-smooth divisor of n. A number n is called k-full, where k is a natural number, if p^k divides n whenever p divides n. We write $d \parallel n$ to indicate that d is a *unitary divisor* of n, i.e., that $d \mid n$ and $\gcd(d, n/d) = 1$.

The Landau–Bachmann o and O-symbols, as well as Vinogradov's $\ll$ notation, are employed with their usual meanings. *Implied constants are absolute unless otherwise specified.*

2. The lower bound: Proof of Theorem 1.1

If p is an odd prime, define the *prime tree* $\mathscr{T}^{(+)}(p)$ associated to p as follows: The root node is p, and for each node q, its child nodes are labeled with the odd prime divisors of $q + 1$. The case of $p = 83621$ is illustrated in Figure 1.

With $q - 1$ replacing $q + 1$, such trees were introduced by Pratt [26] (see also §4). A comprehensive study of such objects has recently been undertaken by Ford, Konyagin, and Luca [11]. For our purposes, the following modest result suffices (cf. [26, p. 217]). Let $f^{(+)}(p)$ denote the total number of nodes in $\mathscr{T}^{(+)}(p)$ (e.g., $f^{(+)}(83621) = 9$).

Lemma 2.1. *For each odd prime p, we have $f^{(+)}(p) \le 2\log p$.*

Proof. We have $f^{(+)}(3) = 1$, so the lemma holds when $p = 3$. Now suppose $q \ge 5$ is prime and that the upper-bound of the lemma holds for all odd primes $p < q$. Then

$$\begin{aligned} f^{(+)}(q) &= 1 + \sum_{\substack{p \mid q+1 \\ p>2}} f^{(+)}(p) \le 1 + 2 \sum_{\substack{p \mid q+1 \\ p>2}} \log p \\ &\le 1 + 2\log\frac{q+1}{2} = 2\log\frac{\mathrm{e}^{1/2}(q+1)}{2} \le 2\log q. \end{aligned}$$ □

We now introduce an algorithm for constructing prime-perfect numbers, given an even, prime-abundant input (cf. the proof of [4, Theorem 4]).

Algorithm A:

Input: An even prime-abundant number n_0
Output: A prime-perfect n for which $n_0 \parallel n$ and n/n_0 is squarefree

```
n ⟵ n_0                                  // Initialize
while rad(σ(n)) ∤ n                       // Loop until prime-perfect
do
    Q ⟵ ∏_{q|σ(n), q∤n} q
    n ⟵ nQ
end
return n
```

Proof of correctness of Algorithm A. We are given that $n = n_0$ satisfies $\mathrm{rad}(n) \mid \sigma(n)$. By the choice of Q, this property is preserved by execution of the `while` loop. So it is enough to show that the algorithm terminates, for then the output n satisfies both $\mathrm{rad}(n) \mid \sigma(n)$ and $\mathrm{rad}(\sigma(n)) \mid n$. Hence, n is prime-perfect. Clearly also $n_0 \parallel n$ and n/n_0 is squarefree.

If n is not already prime-perfect, let Q_0 be the product of the primes dividing $\sigma(n)$ and not n. (So Q_0 is the value of Q when the `while` loop is first executed.) Then at each future execution of the `while` loop, the new primes introduced in Q belong to $\cup_{q \mid Q_0} \mathscr{T}^{(+)}(q)$. (Here we identify $\mathscr{T}^{(+)}(q)$ with the set of primes used to label its nodes.) Since each of the trees $\mathscr{T}^{(+)}(q)$ is finite, the algorithm terminates. □

Proof of Theorem 1.1. Let y be a large real number, and let $m = 2 \cdot 3 \cdot 5 \cdot \cdots$ be the largest product of an initial segment of primes for which $m \le y$. By the prime number theorem, $\omega(m) \sim \log y/\log_2 y$ as $y \to \infty$, and so

$$d(m) = 2^{\omega(m)} = y^{(\log 2+o(1))/\log_2 y} \qquad (y \to \infty).$$

Let ℓ range over all numbers of the form

$$\ell = \prod_{p \mid 2^m-1} p^{e_p}, \quad \text{where each } e_p \in \{3, 5\}.$$

For each such ℓ, consider $n_0 := 2^{m-1}\ell$. Then $2 \mid \sigma(\ell)$ and $\mathrm{rad}(\ell) \mid \sigma(2^{m-1})$, so that n_0 is prime-abundant. The number of n_0 that arise in this way is

$$2^{\omega(2^m-1)} \geq 2^{d(m)-2} = \exp(y^{(\log 2+o(1))/\log_2 y}),$$

as $y \to \infty$. Here the inequality follows from a theorem of Bang [3] that implies that for each $d \mid m$ with $d \notin \{1, 6\}$ there is a prime $p_d \mid 2^m - 1$ with 2 belonging to the exponent d in the integers modulo p_d. Moreover, each n_0 appearing in this construction satisfies

$$n_0 \leq 2^{m-1}(2^m - 1)^5 < 2^{6m}. \tag{2.1}$$

We feed each n_0 into Algorithm A and receive as output a prime-perfect number n for which $n_0 \parallel n$ and n/n_0 is squarefree. Since n_0 is the squarefull part of n, distinct values of n_0 correspond to distinct prime-perfect numbers n.

If Q_0 denotes the product of the primes dividing $\sigma(n_0)$ but not n_0, then, by the proof of correctness of Algorithm A, the output n satisfies

$$n \mid n_0 Q_0 \left(\prod_{p \mid Q_0} \prod_{q \in \mathscr{T}^{(+)}(p)} q \right). \tag{2.2}$$

Crudely, with $f = f^{(+)}$,

$$\prod_{p \mid Q_0} \prod_{q \in \mathscr{T}^{(+)}(p)} q \leq \prod_{p \mid Q_0} p^{f(p)} \leq Q_0^{\sum_{p \mid Q_0} f(p)} \leq Q_0^{2 \log Q_0}, \tag{2.3}$$

by Lemma 2.1. But $Q_0 \leq \sigma(n_0) \leq n_0^2$, and so from (2.2) and (2.3),

$$n \leq n_0 \cdot n_0^2 \cdot \exp(2(\log Q_0)^2) \leq n_0^3 \exp(8(\log n_0)^2).$$

Using (2.1), we see that

$$n \leq 2^{18m} \exp(139m^2) \leq \exp(140m^2),$$

say, once y (and hence m) is large enough.

Setting $X := \exp(140y^2)$, so that $y = \sqrt{(\log X)/140}$, we have shown that the number of prime-perfect numbers contained in $[1, X]$ is at least

$$\exp(y^{(\log 2+o(1))/\log_2 y}) = \exp((\log X)^{(\frac{1}{2}\log 2+o(1))/\log_3 X}),$$

as $y \to \infty$. For large X, we can simply define $y = \sqrt{(\log X)/140}$; then $y = y(X) \to \infty$ as $X \to \infty$, and Theorem 1.1 follows. $\square$

The attentive reader will have noticed that all the prime-perfect numbers constructed here are even. In fact, the second author has made the following conjecture (see [13, B19]):

Conjecture 2.2. *Each prime-perfect number $n > 1$ is even.*

3. The upper bound: Proof of Theorem 1.2

For each natural number n, write $\sigma(n)/n = N/D$, where $N = N(n)$ and $D = D(n)$ are coprime positive integers. The following theorem appears as [22, Theorem 4.1]. Loosely, it says that n is nearly determined by D, the lowest-terms denominator of $\sigma(n)/n$.

Theorem A. *For each $x \geq 1$ and each positive integer d, the number of $n \leq x$ for which $D(n) = d$ is at most $x^{O(1/\sqrt{\log_2 x})}$.*

The next lemma is inspired by Erdős's proof of [7, Theorem 2].

Lemma 3.1. *Given a natural number m, the following algorithm outputs a unitary divisor a of m with $\gcd(a, \sigma(a)) = 1$. Moreover, at most $x^{o(1)}$ inputs $m \leq x$ correspond to the same output a, as $x \to \infty$.*

Algorithm B:

Input: A natural number m
Output: A divisor a of m for which $a \parallel m$ and $\gcd(a, \sigma(a)) = 1$
Factor $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $p_1 > p_2 > \cdots > p_k$.
$a \leftarrow 1$ `// Initialize`
for $i = 1$ **to** k **do** `// Loop over prime power divisors of` m
if $\gcd(\sigma(p_i^{e_i} a), p_i^{e_i} a) = 1$ **then**
| $a \leftarrow p_i^{e_i} a$
end
return a

Remark. Fix a natural number K. We will see from the proof of Lemma 3.1 that if we restrict the input m to K-free numbers, the term $x^{o(1)}$ in the conclusion of Lemma 3.1 can be improved to $x^{O_K(1/\log_2 x)}$.

Proof. It is trivial that the output a of the algorithm is a unitary divisor of m for which $\gcd(a, \sigma(a)) = 1$. So we concentrate on the last half of the lemma. Fix $\epsilon > 0$. We will show that for large x, the number of inputs $m \leq x$ corresponding to a given output a is bounded by x^{ϵ}, uniformly in a. Fix a natural number K with $\frac{1}{K} < \epsilon$.

Suppose that a is the output corresponding to the input $m \leq x$. Write $m = abc$, where c is the K-full part of m/a. If $p \mid bc$, then one of the following two possibilities holds:

(1) $p \mid \sigma(q^e)$, where $q^e \parallel a$, *or*

(2) there is a prime q dividing a with $q > p$ for which $q \mid \sigma(p^e)$, where $p^e \parallel m$.

In case (1) above, $p \mid \sigma(a)$. If $p \mid b$ and is described by case (2), then there is some prime q dividing a for which $x = p$ is a solution in the interval $[0, q)$ to one of the $K-1$ congruences

$$x^e + x^{e-1} + \cdots + x + 1 \equiv 0 \pmod{q}, \quad \text{where} \quad 1 \le e < K.$$

Let $\mathscr{S}$ be the set of all primes p which either divide $\sigma(a)$ or appear as a solution to such a congruence. Then $\mathscr{S}$ depends only on a and contains every prime divisor of b. Since a polynomial of degree e has at most e roots over $\mathbf{Z}/q\mathbf{Z}$, we find that for large x,

$$\#\mathscr{S} \le \omega(\sigma(a)) + \sum_{q \mid a} \sum_{1 \le e < K} e \le \omega(\sigma(a)) + K^2\omega(a).$$

Since $\sigma(a) \le a^2$ and $\omega(h) \ll \frac{\log h}{\log_2 h}$ for all $h \ge 1$, we find that for large x,

$$\#\mathscr{S} \le C \frac{\log x}{\log\log x},$$

where C is a constant depending only on K.

For $i \ge 1$, let p_i denote the ith prime in the natural order. We have just seen that given a, the prime factors of b belong to a prescribed set of size at most $R := \lfloor C \log x / \log\log x \rfloor$. The number of such $b \le x$ is bounded above by the number of $b \le x$ supported on the primes $p_1, \ldots, p_R$, i.e., by $\Psi(x, p_R)$. By the prime number theorem, $p_R \le 2C \log x$ for large x, and now by standard results on smooth numbers (see, e.g., [12, eq. (1.19)]), $\Psi(x, p_R) \le x^{O_K(1/\log_2 x)}$. Since the number of possibilities for c is $\ll x^{1/K}$ (see [10]) and $1/K < \epsilon$, the number of possibilities for $n = abc$, given a, is smaller than x^ϵ for large x. $\square$

The next lemma, which is implicit in the proof of [8, Theorem 4], appears explicitly as [22, Lemma 4.2].

Lemma 3.2. *Let $m \le y$ be a natural number. The number of $n \le y$ for which $\mathrm{rad}(n) \mid m$ is at most $y^{O(1/\log_2 y)}$.*

Remark. We often apply Lemma 3.2 to estimate the number of $n \le y$ with $\mathrm{rad}(n) = m$.

We are now in a position to prove Theorem 1.2. Below, we write $o(1)$ for a quantity that tends to 0 as $x \to \infty$, uniformly in all other parameters.

Proof of Theorem 1.2. Suppose $n \le x$ is prime-abundant. Write $n = AB$, where A is squarefree, B is squarefull, and $\gcd(A, B) = 1$. Write $B = CD$, where C is the output of Algorithm B when $m = B$. Let $L = \lceil \log x \rceil$. Then we may choose $a, b, c \in \{\frac{1}{L}, \frac{2}{L}, \dots, \frac{L-1}{L}, 1\}$ for which

$$A \in [\mathrm{e}^{-1}x^a, x^a], \quad B \in [\mathrm{e}^{-1}x^b, x^b], \quad \text{and} \quad C \in [\mathrm{e}^{-1}x^c, x^c]. \tag{3.1}$$

Since the number of possible triples (a, b, c) is at most $L^3 = x^{o(1)}$, it is enough to prove that the number of prime-abundant $n \in (x/2, x]$ corresponding to a given triple is at most

$$x^{1/3+o(1)}, \tag{3.2}$$

as $x \to \infty$.

First we show that given B, the number of possible values of A is at most $x^{o(1)}$. Since $n = AB$ is prime-abundant, $A \mid \sigma(A)\sigma(B)$. Hence, the lowest-terms denominator of the fraction $\sigma(A)/A$ divides $\sigma(B)$, and so is restricted to $x^{o(1)}$ possible values. (We use here an estimate for the maximal order of the divisor function, such as [15, Theorem 317].) Theorem A now shows that A itself is restricted to a set of size $x^{o(1)}$. Thus, the number of possible values of $n = AB$ is at most

$$x^{b/2+o(1)}, \tag{3.3}$$

since the number of squarefull values of $B \le x^b$ is $\ll x^{b/2}$.

We can easily sharpen this; the number of n in question is bounded above by

$$x^{c/2+o(1)}. \tag{3.4}$$

To see this, notice that C is squarefull, since $C \parallel B$. Thus, there are $\ll x^{c/2}$ choices for C. By Lemma 3.1, C determines B up to $x^{o(1)}$ possibilities. Since B determines A up to $x^{o(1)}$ choices, the number of choices for $n = AB$ is at most $x^{c/2+o(1)}$ also.

By a third and final argument, we show that the number of such n is bounded by

$$x^{a+(b-c)/2+o(1)}. \tag{3.5}$$

The number of choices for A is at most x^a. Also, D is squarefull and $D \ll x^{b-c}$, so that the number of possible values of D is $\ll x^{(b-c)/2}$. So there are at most $x^{a+(b-c)/2+o(1)}$ possibilities for $AD = n/C$. Since n is prime-abundant and $\gcd(C, \sigma(C)) = 1$, we have $\mathrm{rad}(C) \mid \sigma(A)\sigma(D)$. So given A and D, Lemma 3.2 implies there are only $x^{o(1)}$ possibilities for C.

Comparing (3.3), (3.4), and (3.5), we see that the number of prime-abundant $n \in (x/2, x]$ corresponding to the triple (a, b, c) is at most

$$x^{t+o(1)}, \quad \text{where} \quad t = \min\{b/2, c/2, a + (b-c)/2\}.$$

Since $n \gg x^{a+b}$, we have $a + b \le 1 + o(1)$, and so

$$3t = t + t + t \le b/2 + c/2 + a + (b-c)/2 = a + b \le 1 + o(1),$$

whence $t \le 1/3 + o(1)$. This confirms the upper estimate (3.2). $\square$

4. Analogues for Euler's function

In view of the duality between φ and σ, it is natural to wonder about the φ-version of the prime-perfect numbers. Call n *φ-abundant* if every prime dividing n divides $\varphi(n)$, and call n *φ-perfect* if the set of primes dividing n coincides with the set of primes dividing $\varphi(n)$. Let $N_\varphi(x)$ denote the number of φ-perfect $n \le x$, and let $N'_\varphi(x)$ denote the number of φ-abundant $n \le x$.

Since $\varphi(m^2) = m\varphi(m)$, every square is prime-abundant, and so $N'_\varphi(x) \gg x^{1/2}$. (Note the sharp contrast with the result of Theorem 1.2.) More generally, every squarefull number is prime-abundant, which gives a somewhat larger value of the implied constant in this estimate. In our first theorem in this section, we show that $N'_\varphi(x)/x^{1/2}$ tends to infinity, but not too rapidly. Let $L(x)$ be as in the statement of Theorem 1.4.

Theorem 4.1. *For some positive constant c and all large x, we have*

$$N'_\varphi(x) \ge x^{1/2} \exp(c(\log_2 x)^{1/2}/(\log_3 x)^{3/2}).$$

In the opposite direction, we have as $x \to \infty$,

$$N'_\varphi(x) \le x^{1/2} L(x)^{1/2+o(1)}.$$

As a consequence, $N'_\varphi(x) = x^{1/2+o(1)}$.

Proof. We start with the upper bound. By the remark following Lemma 3.2, we can assume that $\operatorname{rad}(n) > x^{1/2}L(x)^{1/2}$. The remaining φ-abundant n satisfy $\gcd(n, \varphi(n)) > x^{1/2}L(x)^{1/2}$; but by [8, Theorem 11],

$$\sum_{m \le x} \gcd(m, \varphi(m)) \le x \cdot L(x)^{1+o(1)},$$

and so the number of such n is at most $x^{1/2}L(x)^{1/2+o(1)}$.

We now turn to the lower bound. We can pick a positive constant c_0 for which the following holds: With $z := c_0 \log_2 x/\log_3 x$ and $P = \prod_{p \le z} p$, the number of $m \le y$ with $P \nmid \varphi(m)$ is $\ll y/\log_2 x$, uniformly for $\sqrt[3]{x} \le y \le x$ (cf. the proof of [20, Lemma 2]). We consider numbers n of the form $n = m^2 A$, where $A \mid P$, $m \le \sqrt{x/A}$, m is coprime to P, and $P \mid \varphi(m)$. Note that each such n is φ-abundant. Moreover, distinct pairs (m, A) give rise to distinct values of n.

It remains to count the number of pairs (m, A). For a given A, the number of $m \le \sqrt{x/A}$ with m coprime to P is $\gg \sqrt{x/A}/\log z \gg \sqrt{x/A}/\log_3 x$, by Mertens' theorem and an elementary inclusion-exclusion argument. Of these m, almost all of them are such that P divides $\varphi(m)$, by our choice of c_0 above. So the number of n we construct is

$$\gg \frac{\sqrt{x}}{\log_3 x} \sum_{A \mid P} \frac{1}{\sqrt{A}} = \frac{\sqrt{x}}{\log_3 x} \prod_{p \le z}\left(1 + \frac{1}{\sqrt{p}}\right) \gg \sqrt{x}(\log_3 x)^{O(1)} \exp\left(\sum_{p \le z} \frac{1}{\sqrt{p}}\right).$$

Since $\sum_{p\le z} p^{-1/2} \sim 2\sqrt{z}/\log z$, we have the lower bound. $\square$

It seems plausible that there are almost as many φ-perfect numbers as φ-abundant numbers, in the sense that

$$N_\varphi(x) = x^{1/2+o(1)}. \tag{4.1}$$

Indeed, (4.1) follows from a standard conjecture, as we now explain. Say that $\eta \in (0,1)$ is *admissible* if there are positive numbers $K = K(\eta)$ and $x_0 = x_0(\eta)$ for which

$$\#\{p \le x : P(p-1) \le x^{1-\eta}\} \ge \frac{x}{(\log x)^K} \qquad (\text{for } x \ge x_0). \tag{4.2}$$

In [6], Erdős used Brun's sieve to show that some η is admissible, and he conjectured that all $\eta < 1$ are admissible. The best unconditional result in this direction is due to Baker and Harman [2], who have shown the admissibility of $\eta = 0.7039$.

Theorem 4.2. *Fix an admissible number η. Then the number of φ-perfect $n \le x$ is at least $x^{\eta/2+o(1)}$, as $x \to \infty$.*

Taking as input the result of Baker and Harman, we obtain the first inequality of Theorem 1.4. If Erdős is right that every $\eta < 1$ is admissible, then Theorems 4.1 and 4.2 give the estimate (4.1). We remark that the Elliott–Halberstam conjecture implies that every $\eta < 1$ is admissible (cf. [1, Theorem 3], [12, §5.1]).

Call m *φ-deficient* if every prime dividing $\varphi(m)$ divides m. Then m is φ-perfect precisely when m is both φ-abundant and φ-deficient.

Lemma 4.3. *Fix an admissible number η. Then the number of φ-deficient $m \le x$ is at least $x^{\eta+o(1)}$, as $x \to \infty$.*

Proof. Let $\alpha = (1-\eta)^{-1}$. Put $z = (\log x/\log\log x)^\alpha$, and put

$$w = x/\exp(2\log x/\log_2 x).$$

By the definition of admissibility, the set $\mathscr{P}$ of primes $p \le z$ for which $P(p-1) \le \log x/\log\log x$ has cardinality at least $(\log x)^\alpha/(\log_2 x)^{O(1)}$; here the O-constant may depend on η. Let $u = \lfloor \frac{\log w}{\log z} \rfloor$, and consider all numbers n that can be formed as a product of u distinct primes from $\mathscr{P}$. Each such n satisfies $n \le w$ and $P(\varphi(n)) \le \log x/\log\log x$. Moreover, as $x \to \infty$, the number of such n is

$$\binom{\#\mathscr{P}}{u} \ge \left(\frac{\#\mathscr{P}}{u}\right)^u \ge x^{\frac{\alpha-1}{\alpha}+o(1)} = x^{\eta+o(1)},$$

by a short computation. For each such n, put

$$m := n \prod_{p \le \log x/\log\log x} p. \tag{4.3}$$

Then

$$m \le w \prod_{p \le \log x/\log\log x} p \le w \exp((1+o(1))\log x/\log\log x) < x$$

for large x, and each such m is φ-deficient. Since distinct values of n give rise to distinct values of m, the result follows. □

Proof of Theorem 4.2. Since $\varphi(m^2) = m\varphi(m)$, the number m^2 is φ-perfect if m is φ-deficient. So Theorem 4.2 follows from Lemma 4.3. □

If one is willing to assume further unproved hypotheses, then one can take the reasoning of Lemma 4.3 and Theorem 4.2 a bit further. It seems plausible that in a wide range of x and y,

$$\frac{\#\{p \le x : p-1 \text{ is } y\text{-smooth}\}}{\pi(x)} \approx \frac{\Psi(x,y)}{x}.$$

It may even be that the left and right-hand sides are asymptotic to one another in the range $x \ge y$ and $y \to \infty$; this is explicitly conjectured in [25], but the thought dates back to [6]. In particular, it seems reasonable to assume the following:

With $\ell = \log_2 x$ and $z = \mathrm{e}^{\ell^2}$, the set $\mathscr{P}$ of primes $p \le z$ with $P(p-1) \le \log x/(2\log_2 x)$ satisfies $\#\mathscr{P} \ge \mathrm{e}^{\ell^2-(1+o(1))\ell\log\ell}$.

Let $w := x/\exp(2\log x/\log_2 x)$. Then whenever m is a product of $k := \lfloor \frac{\log w}{\log z} \rfloor$ primes from $\mathscr{P}$, the number $n := m\prod_{p\le \log x/\log_2 x} p$ is φ-deficient and belongs to $[1,x]$. A quick calculation shows that the number of values of n we have just constructed is $x/L(x)^{1+o(1)}$. Since each n^2 is φ-perfect, this implies that $N_\varphi(x) \ge x^{1/2}/L(x)^{1/2+o(1)}$.

As we show in the rest of this section, for φ-perfect numbers which are squarefull, this (conditional) lower bound is best-possible.

Theorem 4.4. *As $x \to \infty$, the number of φ-perfect $n \le x$ which are squarefull is at most $x^{1/2}/L(x)^{1/2+o(1)}$. Also, as asserted in Theorem 1.4, $N_\varphi(x) \le x^{1/2}/L(x)^{1/4+o(1)}$.*

We do not know if the exponent $\frac{1}{4}$ in the latter half of Theorem 4.4 is optimal; perhaps squares tell almost the whole story and the "correct" exponent is $\frac{1}{2}$.

For the proof of Theorem 4.4, we require the following analogue of Lemma 3.1:

Lemma 4.5. *We can exhibit an algorithm which, given a squarefree number m, outputs a divisor a of m with $\gcd(a,\varphi(a)) = 1$, and which is nearly one-to-one in the following sense: Each output corresponds to at most $x^{O(1/\log_2 x)}$ inputs in $[1,x]$.*

Proof. List the primes dividing m in decreasing order, say $p_1 > p_2 > \cdots > p_k$. Let $a = 1$, and for $1 \le i \le k$, replace a with ap_i if $\gcd(ap_i, \varphi(ap_i)) = 1$. At the end of the algorithm, write $m = ab$. Clearly $(a, \varphi(a)) = 1$. If $p \mid b$, then there must be a prime q dividing a for which $p \mid q-1$; hence $b \mid \varphi(a)$. So the result follows from the maximal order of the divisor function. □

For primes p, define the *Pratt prime tree* $\mathscr{T}^{(-)}(p)$ as follows: The root node is p, and for each node q, its child nodes are labeled with the prime divisors of $q-1$.

Lemma 4.6. *Suppose that n is φ-perfect, and write $n = AB$, where A is squarefree, B is squarefull, and $\gcd(A, B) = 1$. Then A is the product of all those primes not dividing B which appear in at least one of the trees $\mathscr{T}^{(-)}(p)$, for p dividing B. In particular, n is determined entirely by B.*

Proof. Suppose that p divides B. Then $\varphi(p) \mid \varphi(B) \mid \varphi(n)$. Since n is prime-perfect, every prime q dividing $p-1 = \varphi(p)$ divides n. For each such q, we have $q-1 = \varphi(q) \mid \varphi(n)$, and so each prime r dividing $q-1$ divides n. Continuing this process, we see that n is divisible by all the primes in all the trees $\mathscr{T}^{(-)}(p)$, and hence A is divisible by the product of primes appearing in the lemma statement.

Now we show every prime dividing A belongs to some $\mathscr{T}^{(-)}(p)$, where $p \mid B$. Suppose otherwise, and let q be the largest counterexample. Since n is prime-perfect, $q \mid \varphi(A)\varphi(B)$. If $q \mid \varphi(A)$, then $q \mid r-1$ for some prime $r > q$; by the maximality of q, it follows that r belongs to one of the trees $\mathscr{T}^{(-)}(p)$, where $p \mid B$. But then q belongs to $\mathscr{T}^{(-)}(p)$. This contradiction shows that $q \mid \varphi(B)$. Since A and B are relatively prime, $q \nmid B$, and so $q \mid p-1$ for some prime p dividing B. But in this case, q belongs to $\mathscr{T}^{(-)}(p)$. □

Finally, we quote an estimate from [25] concerning the multiplicities of values of the Euler function. Let $\varphi^{-1}(m) = \{n : \varphi(n) = m\}$.

Theorem B. *As $m \to \infty$, we have $\#\varphi^{-1}(m) \le m/L(m)^{1+o(1)}$.*

Proof of Theorem 4.4. For each φ-perfect number $n \le x$, write $n = AB$, with A and B as in Lemma 4.6. It is enough to prove that given A, the number of corresponding values of B is bounded by $x^{1/2}A^{-1/2}L(x)^{-1/2+o(1)}$, uniformly for $A \le L(x)^{1/2}$. Indeed, if this claim is proved, the first assertion of Theorem 4.4 follows immediately upon taking $A = 1$. To obtain the bound on $N_\varphi(x)$, we take two cases: The number of n corresponding to values of $A \le L(x)^{1/2}$ is at most

$$x^{1/2}L(x)^{-1/2+o(1)}\sum_A A^{-1/2} = x^{1/2}/L(x)^{1/4+o(1)},$$

as desired. On the other hand, if $A > L(x)^{1/2}$, then $B \le x/L(x)^{1/2}$, and so the number of possible values of B is $\ll x^{1/2}/L(x)^{1/4}$. Since B determines A by Lemma 4.6, we obtain the stated upper bound on $N_\varphi(x)$.

It remains to prove the initial claim. Fix A. Write $R = \mathrm{rad}(B)$, and notice that $R \le x^{1/2}A^{-1/2}$. Since R determines B in at most $L(x)^{o(1)}$ ways by Lemma 3.2, and B determines A, it is enough to prove that the number of possibilities for R is bounded by $x^{1/2}A^{-1/2}L(x)^{-1/2+o(1)}$. Let d be the output of the algorithm of Lemma 4.5 when $m = R$. That lemma allows us to assume that

$$x^{1/2}A^{-1/2} \ge d > x^{1/2}A^{-1/2}L(x)^{-1/2}. \tag{4.4}$$

Since n is prime-perfect, $\mathrm{rad}(\varphi(d)) \mid AB$, and so if we put

$$\ell(d) = \frac{\mathrm{rad}(\varphi(d))}{\gcd(\mathrm{rad}(\varphi(d)), A)},$$

then $\ell(d) \mid R$. Since d and $\ell(d)$ are coprime, the number of possible values of R is at most

$$x^{1/2}A^{-1/2}\sum_{d}\frac{1}{d\cdot\ell(d)}, \tag{4.5}$$

where the sum is over d satisfying (4.4). Rewrite the sum in the form

$$\sum_{e\le x^{1/2}}\frac{1}{e}\sum_{\substack{d\colon\,\ell(d)=e\\ \text{(4.4) holds}}}\frac{1}{d}. \tag{4.6}$$

We estimate the inner sum by partial summation. Fix $e \le x^{1/2}$. Note that if $\ell(d) = e$, then $\mathrm{rad}(\varphi(d)) \mid Ae$, and so the number of possible values of $\varphi(d)$, given A and e, is bounded by $L(x)^{o(1)}$ (by Lemma 3.2). It now follows from Theorem B that

$$\begin{aligned} G(t) &:= \#\{d \le t : \ell(d) = e\} \\ &\le L(x)^{o(1)}\cdot t/L(t)^{1+o(1)} \quad (\text{as } x\to\infty), \end{aligned}$$

uniformly for $e \le x^{1/2}$ and $t \in [x^{1/2}A^{-1/2}L(x)^{-1/2}, x^{1/2}A^{-1/2}]$. So the inner sum in (4.6) is at most

$$\begin{aligned} \int_{x^{1/2}A^{-1/2}L(x)^{-1/2}}^{x^{1/2}A^{-1/2}}\frac{dG(t)}{t} &\le \frac{G(x^{1/2}A^{-1/2})}{x^{1/2}A^{-1/2}} + \int_{x^{1/2}A^{-1/2}L(x)^{-1/2}}^{x^{1/2}A^{-1/2}}\frac{G(t)}{t^2}\,dt \\ &\le \frac{1}{L(x)^{1/2+o(1)}}, \end{aligned}$$

as $x \to \infty$. Substituting into (4.6) and then (4.5), we obtain the claimed upper bound on the number of possibilities for R. □

5. Problems from the literature

5.1. H-perfect numbers

Harborth [14] has considered another variant of the perfect numbers for which the set of examples is provably infinite. If n is a natural number, let $\mathscr{S}$ range over all possible subsets of divisors of n, and put

$$S(n) = \sum_{\mathscr{S}} \sum_{d \in \mathscr{S}} d.$$

Observe that every divisor d of n occurs in precisely $2^{d(n)-1}$ subsets $\mathscr{S}$, so that $S(n) = \sigma(n) \cdot 2^{d(n)-1}$. We will say n is *H-perfect* if $n \mid S(n)$. (So, e.g., the H-perfects include all multiply perfect numbers.) Harborth showed that the number of H-perfect $n \leq x$ exceeds

$$\log x \cdot \frac{\log\log x}{2\log 2}, \tag{5.1}$$

but remarks that

> Eine vernünftige Abschätzung nach oben scheint sich nicht so einfach zu ergeben.[1]

Our purpose here is to show that Harborth's lower bound may be considerably strengthened and to establish "eine vernünftige Abschätzung nach oben". We begin with a simple characterization of H-perfect numbers. As before, we write $\sigma(n)/n = N/D$, where $N(n)$ and $D(n)$ are coprime natural numbers.

Lemma 5.1. *The natural number n is H-perfect if and only if $D(n)$ is a power of 2.*

Proof. Suppose that n is H-perfect. Then n divides $S(n) = \sigma(n) \cdot 2^{d(n)-1}$, and so $D(n) = n/\gcd(n, \sigma(n))$ divides $2^{d(n)-1}$. Thus, $D(n)$ is a power of 2. Conversely, suppose that $D(n) = 2^t$ for some integer $t \geq 0$. Since $D(n)$ divides n, we have $t \leq \sum_{p^\ell \mid n} 1 \leq d(n) - 1$. Thus, $D(n) = 2^t \mid 2^{d(n)-1}$ and

$$n = \gcd(n, \sigma(n)) \cdot D(n) \mid \sigma(n) \cdot 2^{d(n)-1} = S(n),$$

so that n is H-perfect. $\square$

The following lower bound substantially strengthens (5.1).

Proposition 5.2. *As $x \to \infty$, the number of $n \leq x$ which are H-perfect is at least*

$$\exp((\log x)^{(\log 2 + o(1))/\log_3 x}).$$

[1] A reasonable upper bound does not seem so easy to obtain.

Proof. Let $y \to \infty$, and choose $m \le y$ so that $d(m)$ is maximal. Consider numbers $n = 2^{m-1}\ell$, where ℓ runs over all divisors of $2^m - 1$. Then

$$\frac{\sigma(n)}{n} = \frac{(\sigma(2^{m-1})/\ell)\sigma(\ell)}{2^{m-1}},$$

so that $D(n) \mid 2^{m-1}$, and hence $D(n)$ is a power of 2. So n is H-perfect. Moreover, the number of such n is at least

$$d(2^m - 1) \ge 2^{\omega(2^m-1)} \ge \frac{1}{4}2^{d(m)} \ge \exp(y^{(\log 2+o(1))/\log_2 y}),$$

and each such n belongs to $[1, 2^{2y}]$. (Above, we use the inequality $\omega(2^m - 1) \ge d(m) - 2$, which already featured in the proof of Theorem 1.1.) As in the proof of Theorem 1.1, solving for y in terms of $X := 2^{2y}$ completes the argument. □

The upper bound is more straightforward. Indeed, since $D(n)$ must be a power of 2 and only $O(\log x)$ such powers appear below x, Theorem A immediately gives the following:

Proposition 5.3. *For $x \ge 1$, the number of H-perfect $n \le x$ is bounded by $x^{O(1/\sqrt{\log_2 x})}$.*

5.2. Harmonic and superharmonic numbers

We make some remarks about harmonic numbers, first studied by Ore [21] (and named "harmonic" in [23]), and superharmonic numbers, recently introduced by Cohen [4]. The natural number n is said to be *harmonic* if the harmonic mean of its divisors is an integer. By a short calculation, n is harmonic precisely when $\sigma(n) \mid n \cdot d(n)$.

The distribution of harmonic numbers was studied by Kanold [17], who showed that the count of such numbers in $[1, x]$ is bounded by $x^{1/2+o(1)}$, as $x \to \infty$. This can be easily improved by using a theorem of Wirsing [29], which we quote as Theorem C.

Theorem C. *Let $x \ge 1$, and let α be a positive rational number. The number of $n \le x$ with $\sigma(n)/n = \alpha$ is bounded by $x^{O(1/\log_2 x)}$, uniformly in α.*

Proposition 5.4. *The number of harmonic $n \le x$ is bounded by $x^{O(1/\log_2 x)}$.*

Proof. Suppose that n is harmonic. Put $k = n \cdot d(n)/\sigma(n)$, so that $\sigma(n)/n = d(n)/k$. Since $\sigma(n)/n \ge 1$, we have

$$k \le d(n) \le \max_{m \le x} d(m) \le x^{O(1/\log_2 x)}.$$

(Here we again use [15, Theorem 317].) Thus, the fraction $\sigma(n)/n$ is restricted to $x^{O(1/\log_2 x)}$ possible values. But by Theorem C, each value corresponds to at most $x^{O(1/\log_2 x)}$ possibilities for n. □

Cohen [4] calls n *superharmonic* if $\sigma(n) \mid n^k \cdot d(n)$ for some natural number k. While it is not known whether or not there are infinitely many harmonic numbers, Cohen observes [4, Corollary 3] that there are infinitely many superharmonic numbers n. In fact, using an algorithm similar to our Algorithm A, he proves [4, Theorem 4] that for any N, there is a superharmonic number n for which $N \mid n$.

Call a number n *prime-deficient* if every prime dividing $\sigma(n)$ divides n. (Numbers of this kind for which $\omega(n)$ is bounded have been studied by Luca [19].) We can treat the prime-deficients by a modification of the argument offered for Lemma 4.3: Replace $p-1$ with $p+1$ in the equation (4.2) defining admissibility, and replace $\mathscr{P}$ with the set of primes $p \in (\log x, z]$ with $P(p+1) \leq \log x/\log\log x$. The proof of Lemma 4.3 gives that if η is admissible, then there are at least $x^{\eta+o(1)}$ prime-deficient values of $n \leq x$. In fact, the heuristic argument following Lemma 4.3 yields that the number of prime-deficient $n \leq x$ should be at least $x/L(x)^{1+o(1)}$. Clearly, every prime-deficient n is superharmonic, so this also serves as a lower bound for the count of superharmonic numbers.

We now demonstrate a matching upper bound. This strengthens [4, Theorem 7], where an upper bound of the form $x/\exp(c(\log x)^{1/3})$ was proved.

Theorem 5.5. *As $x \to \infty$, the number of superharmonic $n \leq x$ is at most $x/L(x)^{1+o(1)}$.*

Proof. We fix $\epsilon > 0$, and we show that for large x, the number of superharmonic $n \leq x$ is at most $x/L(x)^{1-\epsilon}$. Fix a natural number K with $\frac{1}{K} < \frac{\epsilon}{2}$. Write $n = AB$, where A is K-free, B is K-full, and $\gcd(A, B) = 1$. We can assume that $B \leq L(x)$, since the number of exceptional n (even ignoring the superharmonic condition) is at most

$$x \sum_{\substack{B > L(x) \\ B\ K\text{-full}}} B^{-1} < x/L(x)^{1-\epsilon/2},$$

once x is large.

Let d be the output of Algorithm B when $m = A$. We can assume that $d > x/L(x)$. Indeed, since A is K-free, the remark following Lemma 3.1 shows that if $d \leq x/L(x)$, then A belongs to a set of size at most

$$\frac{x}{L(x)} \cdot x^{O_K(1/\log_2 x)} = \frac{x}{L(x)^{1+o(1)}}.$$

Since $B \leq L(x)$ and B is K-full, the number of possibiliites for B is smaller than $L(x)^{\epsilon/2}$ for large x. So the number of possibilities for $n = AB$ with $d \leq x/L(x)$ is at most $x/L(x)^{1-2\epsilon/3}$ for large x, which is negligible.

To count the remaining n, we fix both B and $D := d(n)$. For $n \leq x$, we have $d(n) \leq x^{1/\log_2 x}$ for large x, and so the number of possibilities for D is $L(x)^{o(1)}$. Since $d \parallel A \parallel n$, we see that $\sigma(d) \mid \sigma(n)$. But n is superharmonic, so that $\mathrm{rad}(\sigma(d)) \mid$

ABD, and so defining

$$\ell(d) := \frac{\mathrm{rad}(\sigma(d))}{\gcd(\mathrm{rad}(\sigma(d)), BD)},$$

we have that $\ell(d) \mid A$. Since $\gcd(d, \sigma(d)) = 1$, we see that $d \cdot \ell(d) \mid A$. Since $A \le x$, the number of possibilities for A is at most $x \sum_d \frac{1}{d \cdot \ell(d)}$, where the sum is over $d \in (x/L(x), x]$. A similar sum appeared in the proof of Theorem 4.4. Proceeding as in that argument, and invoking the σ-analogue of Theorem C (which is proved in the same way), we find that the number of possibilities for A is at most $x/L(x)^{1+o(1)}$. Since the number of possibilities for the pair (B, D) is at most $L(x)^{\epsilon/2} L(x)^{o(1)}$, the number of remaining possibilities for n is at most $x/L(x)^{1-\epsilon/2+o(1)}$. □

6. Concluding remarks

Rivera [27] has asked whether any numbers n are simultaneously prime-perfect and φ-perfect. Several examples were subsequently found by Luke Pebody (ibid.); his smallest is $n = 2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13^3 \cdot 17^3 \cdot 29^2 \cdot 31^3 \cdot 37^2 \cdot 67^2$, for which

$$\sigma(n) = 2^{16} \cdot 3^7 \cdot 5^2 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 29 \cdot 31 \cdot 37 \cdot 67^2$$

and

$$\varphi(n) = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 29 \cdot 31^2 \cdot 37 \cdot 67.$$

Probably there are infinitely many such n, but this may be difficult to show. In fact, we do not even see how to obtain infinitely many n for which $\mathrm{rad}(\varphi(n)) = \mathrm{rad}(\sigma(n))$.

We are morally certain that each even squarefree integer belongs to the range of the functions $\mathrm{rad}(\varphi(n))$ and $\mathrm{rad}(\sigma(n))$. In fact, we believe that each appears infinitely often in both ranges, but even the weaker version of the claim appears difficult. We can at least prove that both ranges contain a positive proportion of the squarefree numbers. In fact, this is true even if n is restricted to primes p. To see this, we recall a result of Erdős and Odlyzko [9].

Theorem D. *The set of odd natural numbers k with the property that $k \cdot 2^n + 1$ is prime for some $n \ge 1$ is a set of positive lower density. The same holds for $k \cdot 2^n - 1$.*

For a prime $p = k \cdot 2^n + 1$ as above, $\mathrm{rad}(\varphi(p)) = 2 \cdot \mathrm{rad}(k)$. Thus, our claim about the image of $\mathrm{rad}(\varphi(p))$ is a consequence of the following elementary lemma. Similarly, the claim for $\mathrm{rad}(\sigma(p))$ follows from the lemma and the second half of Theorem D.

Lemma 6.1. *If $\mathscr{A}$ is a set of positive lower density, then $\mathrm{rad}(\mathscr{A}) := \{\mathrm{rad}(a) : a \in \mathscr{A}\}$ also has positive lower density.*

Proof. Fix z so that the set of natural numbers with squarefull part $> z$ has upper density smaller than the lower density of $\mathscr{A}$. Discarding from $\mathscr{A}$ those integers with squarefull part $> z$, we can assume that the squarefull part of each $a \in \mathscr{A}$ belongs to the set $\mathscr{S}$ of squarefull numbers $\leq z$. Put $m := \#\mathscr{S}$.

By hypothesis, there is a number $d > 0$ and a real x_0 so that $\frac{1}{x}\#\mathscr{A} \cap [1, x] \geq d$ for all $x \geq x_0$. We claim that $\mathrm{rad}(\mathscr{A})$ has lower density at least d/m. Let $x \geq x_0$. For some $s \in \mathscr{S}$, the set $\mathscr{A}_s(x)$ of $a \in \mathscr{A} \cap [1, x]$ with squarefull part s has size at least dx/m. The function rad restricted to $\mathscr{A}_s(x)$ is 1–1 and maps elements to new numbers which are not larger. Hence, $\mathrm{rad}(\mathscr{A})$ has at least dx/m elements in $[1, x]$. □

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