Remarks on fibers of the sum-of-divisors function

Paul Pollack

For Professor Helmut Maier on his 60th birthday.

Abstract Let σ denote the usual sum-of-divisors function. We show that every positive real number can be approximated arbitrarily closely by a fraction m/n with $\sigma(m) = \sigma(n)$. This answers in the affirmative a question of Erdős. We also show that for almost all of the elements v of $\sigma(\mathbf{N})$, the members of the fiber $\sigma^{-1}(v)$ all share the same largest prime factor. We describe an application of the second result to the theory of L. E. Dickson's amicable tuples, which are a generalization of the ancient notion of an amicable pair.

1 Introduction

Let $\sigma(n) := \sum_{d|n} d$ be the familiar sum-of-divisors function. In this paper, we record two theorems concerning the fibers of σ. The first of these answers in the affirmative a 1959 question of Erdős (see [5, p. 172]).

Theorem 1. *Let $\beta > 0$. For every $\varepsilon > 0$, one can find integers m and n with $\sigma(m) = \sigma(n)$ and $|\frac{m}{n} - \beta| < \varepsilon$.*

The primary tool in the proof will be the remarkable recent theorem of Yitang Zhang approximating the prime k-tuples conjecture.

Our second result concerns the multiplicative structure of elements belonging to a typical fiber. Building on work of Maier and Pomerance [8], Ford [6] developed an extensive theory of σ-preimages and used it to answer a number of delicate questions about the distribution of φ and σ-values. For example, he showed that the count of σ-values in $[1,x]$ is

Paul Pollack
University of Georgia, Boyd Graduate Studies Building, Athens, GA 30602, USA e-mail: pollack@uga.edu

$$\asymp \frac{x}{\log x}\exp\left(C(\log_3 x-\log_4 x)^2+D\log_3 x-(D+\frac{1}{2}-2C)\log_4 x\right)$$

for certain constants $C \approx 0.8178146$ and $D \approx 2.1769687$, and that precisely the same estimate holds for the φ-function. Here we show how Ford's methods can be adapted to prove the following theorem.

Theorem 2. *For asymptotically* 100% *of the values v in the image of the σ-function, all of the elements of the set $\sigma^{-1}(v)$ share the same largest prime factor.*

"Asymptotically 100%" means that the density of such v, relative to $\sigma(\mathbf{N})$, is 1.

Theorem 2 has an amusing consequence for a problem of L. E. Dickson. Recall that m and n form an *amicable pair* if $\sigma(m) = \sigma(n) = m+n$. Dickson [1] (cf. [2, p. 50]) proposed the following generalization: Say that $n_1,\dots,n_k$ form an *amicable k-tuple* if $\sigma(n_i) = n_1+n_2+\cdots+n_k$ for each $i \in \{1,2,\dots,k\}$. (Below, we refer to v as the *common σ-value* of the amicable tuple.) Dickson gave a handful of examples with $k = 3$. For several others, with k as large as 6, see [10, 15, 9].

The distribution of amicable tuples remains mysterious. For example, we do not know if there are infinitely many amicable tuples, even if all k are considered simultaneously. In the case $k = 2$ (the amicable pair case), some progress has been made. In 1955, Erdős [4] showed that the set of natural numbers belonging to an amicable pair has density zero. This result has been steadily sharpened over the years [3, 12, 13, 14]. However, when $k > 2$, we still do not know if the set of numbers belonging to an amicable k-tuple has density zero. Thus, the following corollary of Theorem 2 seems of some interest.

Corollary 1. *Asymptotically* 0% *of the elements in the range of the σ-function appear as the common σ-value of an amicable tuple.*

Notation

We reserve the letters p and q for primes. We let $P^+(n)$ denote the largest prime factor of n, with the convention that $P^+(1) = 1$. We use $\omega(n)$ for the number of distinct primes dividing n and $\Omega(n)$ for the number of prime factors of n counted with multiplicity. We write $\Omega(n,U,T)$ for the number of primes dividing n with $U < p \le T$, again counted with multiplicity. We write $\log_k$ for the kth iterate of the natural logarithm.

2 Proof of Theorem 1

Let $\mathbf{L}$ be the closure of the set $\{\log\frac{m}{n} : \sigma(m) = \sigma(n)\}$. Theorem 1 amounts to the claim that $\mathbf{L} = \mathbf{R}$. Since $\log\frac{m}{n} = -\log\frac{n}{m}$, the set $\mathbf{L}$ is symmetric about 0, and so it is enough to show that $\mathbf{L}$ contains all nonnegative real numbers.

For any finite set of primes $\mathscr{P}$, define $\mathbf{L}^{\mathscr{P}}$ in the same way as $\mathbf{L}$ but with m and n restricted to be divisible by none of the primes in $\mathscr{P}$. The next lemma is fundamental.

Lemma 1. *There is a natural number K for which the following holds: Let $\mathscr{P}$ be a finite set of primes. Let $\alpha_1 < \cdots < \alpha_K$ be any K real numbers. Then $\alpha_j - \alpha_i \in \mathbf{L}^{\mathscr{P}}$ for some pair of i and j with $1 \le i < j \le K$.*

The proof requires that we recall Dickson's prime k-tuples conjecture and the spectacular recent progress towards it made by Zhang [18]. A collection of linear polynomials $a_1x+b_1, \dots, a_kx+b_k \in \mathbf{Z}[x]$, each with positive leading coefficient, is called *admissible* if $\gcd\{\prod_{i=1}^{k}(a_in+b_i)\}_{n\in\mathbf{Z}} = 1$. Dickson conjectured that for any admissible collection, there are infinitely many n for which all of the a_in+b_i are simultaneously prime.

Zhang's breakthrough result in this direction was the following:

Proposition 1. *There is a natural number K for which the following holds: Suppose that a_1x+b_1, a_2x+b_2, $\dots$, a_Kx+b_K is admissible. For some pair of i and j with $1 \le i < j \le K$, the expressions a_in+b_i and a_jn+b_j simultaneously represent primes for infinitely many natural numbers n.*

Zhang states his result only in the case when all $a_i = 1$. A stronger version of the proposition — explicitly stated for general linear forms — appears in recent work of Maynard (see [11, Theorem 3.1]).

Proof (Lemma 1). Assuming Dickson's conjecture, Schinzel and Sierpiński [16, p. 193] showed that there are solutions to $\sigma(m) = \sigma(n)$ with the ratio m/n arbitrarily large. We modify their approach to demonstrate Lemma 1. Our proof will show that if K is acceptable in Proposition 1, then it is also acceptable in Lemma 1.

The set $\{\log\frac{\sigma(n)}{n} : \gcd(n, \prod_{p\in\mathscr{P}} p) = 1\}$ is dense in $[0,\infty)$. Indeed, if f is any nonnegative additive function for which (1) $\sum_p f(p)$ diverges, and (2) $f(p) \to 0$ along primes p, then the values of $f(n)$, with n squarefree, are dense in $[0,\infty)$. (This follows from a straightforward application of the greedy algorithm.) We apply this general fact with f any additive function having $f(p) = 0$ for $p \in \mathscr{P}$ and $f(p) = \log\frac{\sigma(p)}{p}$ for $p \notin \mathscr{P}$.

We can assume all $\alpha_i > 0$, by replacing each α_i with $\alpha_i + \alpha_0$ for a suitably large α_0. For each $1 \le i \le K$, fix a sequence $\{A_j^{(i)}\}_{j=1}^{\infty}$ of integers coprime to $\prod_{p\in\mathscr{P}} p$ satisfying

$$\lim_{j\to\infty} \log\frac{\sigma(A_j^{(i)})}{A_j^{(i)}} = \alpha_i.$$

For each $j \in \mathbf{N}$, we apply Proposition 1 to the collection $\{\sigma(A_j^{(i)})x - 1\}_{i=1}^{K}$. This is an admissible collection, since the product of the polynomials at $x = 0$ is ± 1. By Proposition 1, we can find a natural number n_j, and integers $1 \le a_j < b_j \le K$, for which $p_j := \sigma(A_j^{(a_j)})n_j - 1$ and $q_j := \sigma(A_j^{(b_j)})n_j - 1$ are simultaneously prime. Moreover, we can assume that p_j and q_j are both larger than j, larger than $A_j^{(1)}, \dots, A_j^{(K)}$, and larger than any element of $\mathscr{P}$. Observe that

$$\frac{p_j+1}{q_j+1} = \frac{\sigma(A_j^{(a_j)})}{\sigma(A_j^{(b_j)})}, \quad \text{so that} \quad \sigma(p_j A_j^{(b_j)}) = \sigma(q_j A_j^{(a_j)});$$

also, both $p_j A_j^{(b_j)}$ and $q_j A_j^{(a_j)}$ are prime to $\prod_{p \in \mathscr{P}} p$. There are only $\binom{K}{2}$ possibilities for the pair (a_j, b_j), and so some choice (a,b) must be taken on for infinitely many j. As $j \to \infty$ through corresponding values,

$$\log \frac{q_j A_j^{(a)}}{p_j A_j^{(b)}} = \log \frac{(q_j+1)A_j^{(a)}}{(p_j+1)A_j^{(b)}} + o(1) = \log \frac{\sigma(A_j^{(b)})}{A_j^{(b)}} - \log \frac{\sigma(A_j^{(a)})}{A_j^{(a)}} + o(1),$$

which gives that $\alpha_b - \alpha_a \in \mathbf{L}^{\mathscr{P}}$.

Proof (Theorem 1). We show that $\mathbf{R}_{\geq 0}$ is contained in $\mathbf{L}$. Take any $\alpha \geq 0$.

Let $\varepsilon > 0$. Apply Lemma 1 with $\mathscr{P} = \emptyset$ and $\alpha_1, \dots, \alpha_k$ chosen as $0, \frac{1}{K}\varepsilon, \dots, \frac{K-1}{K}\varepsilon$. We find that one of $\frac{1}{K}\varepsilon, \frac{2}{K}\varepsilon, \dots, \frac{K-1}{K}\varepsilon \in \mathbf{L}$; in particular, $\mathbf{L} \cap (\frac{\varepsilon}{2K}, \varepsilon) \neq \emptyset$. Thus, we can choose m_1 and n_1 with $\sigma(m_1) = \sigma(n_1)$ and $\frac{\varepsilon}{2K} < \log \frac{m_1}{n_1} < \varepsilon$. Suppose we have already defined m_j and n_j. Apply Lemma 1 with the same $\alpha_1, \dots, \alpha_k$ but with $\mathscr{P}$ the set of primes dividing $\prod_{i=1}^{j} m_i n_i$. We find that there are natural numbers m_{j+1} and n_{j+1} with $\sigma(m_{j+1}) = \sigma(n_{j+1})$, $\gcd(m_{j+1}n_{j+1}, \prod_{i=1}^{j} m_i n_i) = 1$, and $\frac{\varepsilon}{2K} < \log \frac{m_{j+1}}{n_{j+1}} < \varepsilon$. We continue ad infinitum to produce infinite sequences $\{m_j\}$ and $\{n_j\}$.

Since each $\log \frac{m_j}{n_j} \geq \frac{\varepsilon}{2K}$, we may choose J with $\sum_{j=1}^{J} \log \frac{m_j}{n_j} \geq \alpha$. Moreover, if J is chosen minimally, then $\alpha \leq \sum_{j=1}^{J} \log \frac{m_j}{n_j} < \alpha + \varepsilon$. With $m := \prod_{j=1}^{J} m_j$ and $n := \prod_{j=1}^{J} n_j$, we see that $\sigma(m) = \sigma(n)$ and $0 \leq \log \frac{m}{n} - \alpha < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, $\alpha \in \mathbf{L}$.

Remark 1. Ford's methods (see the proof of the lower bound in [6]) show that given any fiber $\sigma^{-1}(v) = \{n_1, \dots, n_k\}$, a positive proportion of all fibers $\sigma^{-1}(w)$ have the form $\{dn_1, \dots, dn_k\}$ for some natural number d. Clearly, dilating by a factor of d does not change the ratios between elements of a set. Thus, not only is every $\beta > 0$ well-approximable by a ratio m/n, where $\sigma(m) = \sigma(n)$, but it is not so unusual to see a ratio close to β. For example, a positive proportion of $v \in \sigma(\mathbf{N})$ have two preimages m and n with $|\frac{m}{n} - \pi| < 10^{-10}$.

Remark 2. It seems interesting to observe that in the statement of Theorem 1, m and n can be taken to be coprime. Indeed, the argument given above produces squarefree integers m and n. Since $\sigma(m) = \sigma(n)$, if we write $m/n = m'/n'$ in lowest terms, then also $\sigma(m') = \sigma(n')$.

Remark 3. With obvious modifications, our argument will show that Theorem 1 also holds with φ replacing σ. In the same source [5] where Erdős mentions the problem for σ, he claims that this φ-variant can be handled in an elementary fashion.

3 Proof of Theorem 2

Overview of the basic method

The proof follows the method of [7] for showing that most φ-values are not σ-values, and vice versa. We review the strategy of that argument here. For $f \in \{\varphi, \sigma\}$, let $V_f(x)$ be the number of f-values belonging to $[1,x]$. As already alluded to in the introduction, one knows [6, Theorem 14] that $V_\varphi(x) \asymp V_\sigma(x)$ for $x \geq 1$; thus, the main result of [7] follows if it is shown that the number of common φ, σ values in $[1,x]$ is $o(V_\varphi(x) + V_\sigma(x))$.

To this end, one begins by constructing [7, §3] sets $\mathscr{A}_\varphi$ and $\mathscr{A}_\sigma$ with the property that almost all f-values in $[1,x]$ have all their f-preimages in $\mathscr{A}_f$. The precise definition of the sets $\mathscr{A}_f$ [7, p. 1679] is quite intricate and incorporates both "anatomical" and "structural" constraints. By "anatomical", we mean multiplicative constraints of the sort that often arise in elementary number theory. For example, we insist that for $a \in \mathscr{A}_f$, neither a nor $f(a)$ has an extraordinarily large squarefull divisor or "too many" prime divisors. Chief among the anatomical constraints is the requirement that every prime p dividing an element of $\mathscr{A}_f$ be a *normal* prime, meaning that the prime divisors of both $p-1$ and $p+1$ are roughly uniformly distributed on a double-logarithmic scale.

By "structural", we mean that extensive use is made of the results of [6] describing the fine structure of typical f-values and their preimages. As an example, precise inequalities are imposed on the prime divisors of elements of $\mathscr{A}_f$; the ordered list of such primes, after a double-logarithmic rescaling, must (up to a small error) correspond to a point in the *fundamental simplex* of [6, §3]. In addition, we require — and this is the main innovation of [7] — that a particular linear combination of renormalized prime factors be slightly less than 1 (this is condition (8) below in the definition of the set $\mathscr{A}_\sigma$). This ensures that sieve bounds (such as those that feature in Lemma 2 below) eventually yield a nontrivial estimate.

Having constructed such sets $\mathscr{A}_f$, it is enough to study how many common φ, σ values appear as solutions to an equation of the form

$$\varphi(a) = \sigma(a'), \quad \text{where} \quad a \in \mathscr{A}_\varphi, a' \in \mathscr{A}_\sigma. \tag{1}$$

Write $a = p_0 p_1 p_2 \cdots$ and $a' = q_0 q_1 q_2 \cdots$, where the sequences of primes p_i and q_j are nonincreasing. The normality condition in the definition of the sets $\mathscr{A}_f$ implies that for small values of i, we have $p_i \approx q_i$, at least on a double logarithmic scale. We classify the primes p_i and q_i dividing a and a' into three categories: "large", "small", and "tiny" (as described in [7, §5A]). Then (1) gives rise to an equation of the form

$$(p_0 - 1)(p_1 - 1) \cdots (p_{k-1} - 1) f d = (q_0 + 1)(q_1 + 1) \cdots (q_{k-1} + 1) e. \tag{2}$$

Here $p_0, \dots, p_{k-1}$ and $q_0, \dots, q_{k-1}$ are the large primes in a and a' (respectively), f is the contribution to $\varphi(a)$ of the small primes, d is the contribution to $\varphi(a)$ of the tiny primes, and e is the contribution to $\sigma(a')$ of both the small and tiny primes.

To finish the argument, we require an estimate for the number of solutions to equations of the form (2). We prove a lemma ([7, Lemma 4.1], cf. Lemma 2 below) counting the number of solutions $p_0, \dots, p_{k-1}, q_0, \dots, q_{k-1}, e, f$ to possible equations of the form (2), given d and given intervals encoding the rough location of the primes p_i and q_i. (The phrase "possible equations" means that there are many further technical hypotheses in the lemma, but that these hypotheses are automatically satisfied because of our choice of the sets $\mathscr{A}_f$.) Finally, we sum the estimate of the lemma over all possible values of d and all possible selections of intervals; this allows us to show [7, p. 1695] that

$$\#\{\varphi(a) : (a,a') \in \mathscr{A}_\varphi \times \mathscr{A}_\sigma \text{ and } \varphi(a) = \sigma(a')\} \ll \frac{x}{\log x} \exp\left(-\frac{1}{4}(\log_2 x)^{1/2}\right),$$

which is $o(V_\varphi(x) + V_\sigma(x))$ with much room to spare.

3.1 Definition of the set $\mathscr{A}_\sigma$

Theorem 2 will be proved by modifying the above procedure. We start by giving a careful definition of the set $\mathscr{A}_\sigma$ appearing above. The set $\mathscr{A}_\varphi$ can be defined in an entirely similar manner, but we will not need this.

Put

$$F(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \text{where each} \quad a_n = \int_n^{n+1} \log t \, dt. \tag{3}$$

The series defines F as a continuous, increasing function of z on $(0,1)$. Moreover, $F(z) \to \infty$ as $z \uparrow 1$. Hence, there is a unique $\rho \in (0,1)$ with $F(\rho) = 1$; numerically, $\rho \approx 0.5426$. We let $C = \frac{1}{2|\log \rho|}$, which is ≈ 0.8178. (We met this constant already in the introduction.)

Given a natural number n, write $n = p_0(n)p_1(n)p_2(n)\dots$, where $p_0(n) \geq p_1(n) \geq p_2(n) \geq \dots$ are the primes dividing n (with multiplicity). Put

$$x_i(n;x) = \begin{cases} \log_2 p_i(n)/\log_2 x & \text{if } i < \Omega(n) \text{ and } p_i(n) > 2, \\ 0 & \text{otherwise.} \end{cases}$$

For each real number L and each $\boldsymbol{\xi} = (\xi_0, \dots, \xi_{L-2})$, we let $\mathscr{S}_L(\boldsymbol{\xi})$ be the set of $(x_1, \dots, x_L) \in \mathbf{R}^L$ with $0 \leq x_L \leq x_{L-1} \leq \cdots \leq x_1 \leq 1$, and satisfying the system of inequalities

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_Lx_L &\leq \xi_0 \\ a_1x_2 + \cdots + a_{L-1}x_L &\leq \xi_1 x_1 \\ &\vdots \\ a_1x_{L-1} + a_2x_L &\leq \xi_{L-2}x_{L-2}. \end{aligned}$$

The region corresponding to $\xi = (1,1,\ldots,1)$ is called the L-dimensional *fundamental simplex*. Let

$$L_0(x) = \lfloor 2C(\log_3 x - \log_4 x) \rfloor;$$

when x is clear from context, we abbreviate $L_0(x)$ to L_0. One of the key observations of [6] is that if the components of $\boldsymbol{\xi}$ are slightly larger than 1, and L is a little smaller than L_0, then the n for which $(x_1(n;x),\ldots,x_L(n;x)) \in \mathscr{S}_L(\boldsymbol{\xi})$ generate almost all σ-values in $[1,x]$. (See condition (5) below in the definition of $\mathscr{A}_\sigma$ for one way of making this precise.)

Let $S \geq 2$. We say that a prime p is *S-normal* if $\Omega(p+1,1,S) \leq 2\log_2 S$ and, for every pair of real numbers (U,T) with $S \leq U < T \leq p+1$,

$$|\Omega(p+1,U,T) - (\log_2 T - \log_2 U)| < \sqrt{\log_2 S \log_2 T}.$$

In other words, there are not exorbitantly many prime divisors of $p+1$ up to S, and the larger prime divisors of $p+1$ are roughly uniformly distributed on a double logarithmic scale. (In [7], the definition of S-normal required the same constraints also on $p-1$, but for this paper working with $p+1$ is sufficient.)

We can now give a precise definition of the set $\mathscr{A}_\sigma$. Fix $\varepsilon \in (0,1/2)$ and assume throughout that $x \geq x_0(\varepsilon)$. Let

$$S = \exp((\log_2 x)^{36}), \quad \omega = (\log_2 x)^{-\frac{1}{2}+\frac{\varepsilon}{2}},$$
$$L = \lfloor L_0 - 2\sqrt{\log_3 x} \rfloor, \quad \text{and} \quad \xi_i = 1 + \frac{1}{10(L_0 - i)^3}.$$

Then $\mathscr{A}_\sigma = \mathscr{A}_\sigma(\varepsilon, x)$ is the set of $n = p_0(n)p_1(n)\ldots$ with $\sigma(n) \leq x$ that satisfy all of

(0) $n \geq x/\log x$,
(1) every squarefull divisor m of n or of $\sigma(n)$ has $m \leq (\log x)^2$,
(2) all of the primes $p_j(n)$ are S-normal,
(3) $\Omega(\sigma(n)) \leq 10\log_2 x$ and $\Omega(n) \leq 10\log_2 x$,
(4) if $d \parallel n$ and $d \geq \exp((\log x)^{1/2})$, then $\Omega(\sigma(d)) \leq 10\log_2 \sigma(d)$,
(5) $(x_1(n;x),\ldots,x_L(n;x)) \in \mathscr{S}_L(\boldsymbol{\xi})$,
(6) n has at least $L+1$ odd prime divisors (counted with multiplicity),
(7) $P^+(p_0+1) \geq x^{1/\log_2 x}$, $p_1(n) < x^{1/(100\log_2 x)}$,
(8) $a_1x_1(n;x) + \cdots + a_Lx_L(n;x) \leq 1 - \omega$.

The following statement appears as [7, Lemma 3.2].

Proposition 2. *The number of σ-values in $[1,x]$ which have a preimage $n \notin \mathscr{A}_\sigma$ is*

$$\ll V_\sigma(x)(\log_2 x)^{-\frac{1}{2}+\varepsilon}.$$

This makes precise our claim in the overview that the elements of $\mathscr{A}_\sigma$ generate almost all σ-values in $[1,x]$.

3.2 *The proof proper*

Say that the σ-value v is *exceptional* if it possesses two preimages a and a' for which $P^+(a) \neq P^+(a')$. We wish to show that the count of exceptional σ-values in $[1,x]$ is $o(V_\sigma(x))$, as $x \to \infty$. In view of Proposition 2, we may assume v is such that all of its preimages belong to $\mathscr{A}_\sigma$. Pick preimages a and a' with $P^+(a) \neq P^+(a')$, and write

$$a = p_0 p_1 p_2 \dots, \quad \text{and} \quad a' = q_0 q_1 q_2 \dots,$$

where p_i and q_i are nonincreasing sequences.

3.2.1 Rewriting $\sigma(a) = \sigma(a')$

Following the overview presented above, our first task is to deduce from the equation $\sigma(p_0 p_1 p_2 \cdots) = \sigma(q_0 q_1 q_2 \cdots)$ an auxiliary equation of the shape

$$(p_0+1)(p_1+1)\cdots(p_{k-1}+1)fd = (q_0+1)(q_1+1)\cdots(q_{k-1}+1)e. \tag{4}$$

We select k — our cutoff between “large” and “small” primes — in exact parallel with how k is selected in [7, §5A]. In other words, we choose k_0 as the smallest index for which

$$\log_2 P^+(p_{k_0}+1) \leq (\log_2 x)^{1/2+\varepsilon/10},$$

and we take $k = k_0$ unless p_{k_0} and p_{k_0-1} are too close together in a certain technical sense, in which case we take $k = k_0 - 1$.[1] As explained on [7, p. 1689], the properties of $\mathscr{A}_\sigma$ imply that

$$k \sim (1/2 - \varepsilon/10)L. \tag{5}$$

(This asymptotic formula is essentially [7, Lemma 5.3]; one has only to change the $p-1$ to a $p+1$ in its proof.) Moreover, if $0 \leq i < k$, then

$$\log_2 p_i > (\log_2 x)^{1/2+\varepsilon/10} \quad \text{and} \quad \log_2 q_i > (\log_2 x)^{1/2+\varepsilon/11}.$$

(For this, see again [7, p. 1689].) These last estimates, along with condition (1) in the definition of $\mathscr{A}_\sigma$, guarantee that $p_i^2 \nmid a$ and $q_i^2 \nmid a'$. Thus, the first k factors on the left and right-hand sides of (4) represent the contribution to $\sigma(a)$ and $\sigma(a')$ from the “large” primes $p_0, \dots, p_{k-1}$ and $q_0, \dots, q_{k-1}$, respectively.

To make the right-hand side of (4) coincide with $\sigma(a')$, it suffices to define

$$e = \sigma(q_k q_{k+1} q_{k+2} \cdots).$$

The choices of f and d are slightly more delicate. If $p_{L-1} \neq p_L$, then we put

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[1] Precisely: With $\eta := 10L\sqrt{\log_2 S/\log_2 x}$, we choose $k = k_0$ unless $x_{k_0-1}(n;x) - x_{k_0}(n;x) < 20\eta$. This becomes relevant for verifying that the intervals $[u_i, v_i]$ selected later in the proof satisfy the conditions of Lemma 2 below. Since we will refer to [7] for the selection of u_i and v_i and this verification, it does not make sense here to go into more detail.

$$f = \sigma(p_k p_{k+1} \cdots p_{L-1}), \quad d = \sigma(p_L p_{L+1} \cdots).$$

(In the language of the overview, L is what one thinks of as the cutoff between small and tiny primes.) In the general case, we let A be the largest divisor of a supported on the primes $p_k, \ldots, p_{L-1}$, and we put $f = \sigma(A)$ and $d = \sigma(a/(p_1 \cdots p_{k-1}A))$. Then (4) holds. Note that by assumption, $p_0 \neq q_0$.

3.2.2 The key sieve lemma

To continue, we need a tool that allows us to count solutions to (4). We use the following variant of [7, Lemma 4.1], which was proved by repeated application of the upper bound sieve. As the required changes to the proof of [7, Lemma 4.1] are little more than typographical, we omit the demonstration.

Lemma 2. *Let y be large, $k \geq 1$, $30 \leq S \leq v_k \leq v_{k-1} \leq \ldots \leq v_0 = y$, and $u_j \leq v_j$ for $0 \leq j \leq k-1$. Suppose that $1 \leq B \leq y^{1/10}$, and put $\delta = \sqrt{\log_2 S/\log_2 y}$. Set $\nu_j = \log_2 v_j/\log_2 y$ and $\mu_j = \log_2 u_j/\log_2 y$. Suppose that d is a natural number for which $P^+(d) \leq v_k$. Moreover, suppose that both of the following hold:*

(a) For $2 \leq j \leq k-1$, either $(\mu_j, \nu_j) = (\mu_{j-1}, \nu_{j-1})$ or $\nu_j \leq \mu_{j-1} - 2\delta$. Also, $\nu_k \leq \mu_{k-1} - 2\delta$.
(b) For $1 \leq j \leq k-2$, we have $\nu_j > \nu_{j+2}$.

The number of solutions of

$$(p_0+1)\cdots(p_{k-1}+1)fd = (q_0+1)\cdots(q_{k-1}+1)e \leq y/B,$$

in $p_0, \ldots, p_{k-1}, q_0, \ldots, q_{k-1}, e, f$ satisfying

(i) $\gcd(\prod_{i=0}^{k-1} p_i, \prod_{j=0}^{k-1} q_j) = 1$*;*
(ii) p_i and q_i are S-normal primes;
(iii) $u_i \leq P^+(p_i+1), P^+(q_i+1) \leq v_i$ for $0 \leq i \leq k-1$;
(iv) no p_i+1 or q_i+1 is divisible by r^2 for a prime $r \geq v_k$;
(v) $P^+(ef) \leq v_k$; $\Omega(f) \leq 4l \log_2 v_k$;
(vi) p_0+1 has a divisor $\geq y^{1/2}$ which is composed of primes $\geq v_1$;

is

$$\ll \frac{y}{dB}(c\log_2 y)^{6k}(k+1)^{\Omega(d)}(\log v_k)^{8(k+l)\log(k+l)+1}(\log y)^{-2+\sum_{i=1}^{k-1} a_i\nu_i+E},$$

where $E = \delta\sum_{i=2}^{k}(i\log i + i) + 2\sum_{i=1}^{k-1}(\nu_i - \mu_i)$. Here c is an absolute positive constant, and the a_i are as defined in (3).

Remark 4. The condition (i) is not present in [7, Lemma 4.1]. The explanation is that applying the upper bound sieve requires a linear independence condition on the linear forms. This condition is automatic when treating the equation (2), because the left-hand shifted prime factors are shifted by -1 whereas the right-hand shifts are by $+1$. In (4), the primes are shifted by $+1$ on both sides, forcing us to assume (i).

3.2.3 Capturing solutions with Lemma 2: Attempt #1

Given a pair $(a,a') \in \mathscr{A}_\sigma \times \mathscr{A}_\sigma$ satisfying

$$\sigma(a) = \sigma(a') \quad \text{and} \quad P^+(a) \neq P^+(a'), \tag{6}$$

we described in §3.2.1 how to construct a solution to (4). By a 'solution', we mean the values of $p_0, \dots, p_{k-1}$, $q_0, \dots, q_{k-1}$, d, e, and f. On the other hand, given a solution to (4) that arose this way, we can recover the common value of $\sigma(a)$ and $\sigma(a')$ by computing either side of (4). So we can bound the number of exceptional σ-values having all preimages in $\mathscr{A}_\sigma$ by the number of solutions to (4) that arise — in the manner detailed in §3.2.1 — from $(a,a') \in \mathscr{A}_\sigma \times \mathscr{A}_\sigma$ satisfying (6). Henceforth, when we speak of a 'solution' to (4), we always mean a solution that arose this way.

We group solutions to (4) according to the value of k, the value of d, and the "rough positions" of the primes $P^+(p_i+1)$ and $P^+(q_i+1)$. (In view of the fact that each p_i, q_i is normal, this is essentially the same as grouping by the positions of the p_i and q_i themselves, but turns out to be technically more convenient.) Our hope is to apply Lemma 2 to bound the number of solutions in each group, and then sum over all the groups.

Suppose we start with a solution to (4) and want to place it in a group. What does it mean precisely to specify "the rough positions" of the primes $P^+(p_i+1)$ and $P^+(q_i+1)$? We will interpret this to mean that we specify $u_0, \dots, u_{k-1}$ and $v_0, \dots, v_k$ so that, taking $y := x$,

- $30 \leq S \leq v_k \leq v_{k-1} \leq \dots \leq v_0 = y$
- $u_j \leq v_j$ for $0 \leq j \leq k-1$,
- (a) and (b) in Lemma 2 hold,
- (iii) holds.

A systematic way of choosing u_i and v_i to satisfy these criteria is described in detail in [7, §5B].[2]

Moreover, if one selects the u_i and v_i by that recipe, and takes

$$B = 1 \quad \text{and} \quad l = L - k,$$

then our solution to (4) satisfies not only (iii) but in fact every condition of Lemma 2 except possibly condition (i). That is, $P^+(d) \leq v_k$ and all of (ii)–(vi) hold. This follows mutatis mutandis from the corresponding proofs in [7, §5C]. The only point that merits further discussion is the verification that $P^+(ef) \leq v_k$ (as claimed in (v)) and the related point that $P^+(d) \leq v_k$. Here the more wild behavior of σ on prime powers, vis-à-vis φ, complicates matters.

Let $r := P^+(e)$. Since $k < L$ and a' has at least $L+1$ distinct odd prime divisors, $e = \sigma(q_k q_{k+1} \cdots) > 1$ and so $r > 1$. Choose a prime power R exactly dividing $q_k q_{k+1} q_{k+2} \cdots$ for which $r \mid \sigma(R)$. If R is a proper prime power, then (1) in the

[2] Since we are working with solutions to (4) instead of (2), one should read [7, §5B] mentally replacing each expression of the form $p-1$ with $p+1$.

definition of $\mathscr{A}_\sigma$ implies that $R \le (\log x)^2$ and so

$$r \le \sigma(R) < 2(\log x)^2 < S \le v_k.$$

So we can assume that R is a prime divisor of $q_k q_{k+1} \cdots$. Then $r \mid R+1$, and $r \le P^+(R+1) \le \max\{3,R\} \le q_k$. From the fifth display on [7, p. 1692],

$$\log_2 q_k/\log_2 x \le \log_2 p_k/\log_2 x + (2k+1)\sqrt{\log_2 S/\log_2 x} \le \log_2 v_k/\log_2 x.$$

Thus, $r \le q_k \le v_k$. An entirely similar argument shows that $P^+(\sigma(p_k p_{k+1}\cdots)) \le v_k$, so that both $P^+(f) \le v_k$ and $P^+(d) \le v_k$.

If (i) were to always hold, it would be clear how we ought to finish the proof of Theorem 2. In that case, every solution to (4) would fit in a group of the sort counted by Lemma 2. Summing the estimate of the lemma over the possible k, d, and u_i, v_i (that is, over all possible groups of solutions) would give us an upper bound on the count of all solutions. However, there is no reason for (i) to always hold. It could well be that the list $p_1,\dots,p_{k-1}$ overlaps with the list $q_1,\dots,q_{k-1}$. So we must work a bit harder before Lemma 2 can be applied.

3.2.4 Attempt #2

There is an easy fix for the problem we have just run into: Do not attempt to apply Lemma 2 until after canceling factors arising from the common large primes! Given a solution to (4), put

$$m = \gcd(p_0 \cdots p_{k-1}, q_0 \cdots q_{k-1}).$$

By assumption, $p_0 \ne q_0$. It follows that neither p_0 nor q_0 can divide m. Indeed, from conditions (3) and (7) in our definition of $\mathscr{A}_\sigma$, both $p_0, q_0 > x^{1/2}$ while each $p_i, q_i \le x^{\frac{1}{100\log_2 x}}$ for $i \ge 1$.

For each prime p dividing m, cancel the factors of $p+1$ from both sides of (4). Relabeling, we obtain an equation of the form

$$(\tilde{p}_0+1)\cdots(\tilde{p}_{K-1}+1)df = (\tilde{q}_0+1)\cdots(\tilde{q}_{K-1}+1)e \tag{7}$$

where $K = k-\omega(m)$ and the common value of both sides of (7) is at most $x/\sigma(m)$. We may assume that the $\tilde{p}_i$ and $\tilde{q}_i$ are in nonincreasing order. Since $\gcd(m, p_0q_0) = 1$, we have $K \ge 1$, $\tilde{p}_0 = p_0$, and $\tilde{q}_0 = q_0$. Write each

$$\tilde{p}_i = p_{j_i}, \quad \text{and} \quad \tilde{q}_i = q_{j'_i},$$

where the indices i and i' satisfy $i \le j_i, j'_i < k$ for $0 \le i < K$.

In the last section, our choices of parameters in Lemma 2 possibly failed to capture the solution $p_0,\dots,p_{k-1}$, $q_0,\dots,q_{k-1}$, d,e,f to (4). We now describe how to capture the solution $\tilde{p}_0, \dots, \tilde{p}_{K-1}$, $\tilde{q}_0, \dots, \tilde{q}_{K-1}$, d, e, f to (7) by making slightly different choices of these parameters.

We continue to assume that u_i and v_i are chosen as in the preceding section. Since hypothesis (a) of Lemma 2 holds, for $i \geq 1$ the intervals $[u_i, v_i]$ and $[u_{i+1}, v_{i+1}]$ either coincide or are disjoint. Now appealing to (iii) — which was also satisfied for our choices of u_i, v_i — we see that

$$u_{j_i} = u_{j_i'} \leq P^+(p_{j_i}+1), P^+(q_{j_i'}+1) \leq v_{j_i} = v_{j_i'}$$

for every $1 \leq i \leq K-1$. Put

$$\tilde{u}_i = u_{j_i} \text{ and } \tilde{v}_i = v_{j_i} \text{ for } 0 \leq i \leq K-1, \quad \text{and put} \quad \tilde{\nu}_K = \nu_k.$$

From the second half of condition (7) in the definition of $\mathscr{A}_\sigma$ and the estimate $k < L = O(\log_3 x)$,

$$\sigma(m) \leq m^2 \leq (p_1 \cdots p_{k-1})^2 \leq x^{O(\log_3 x/\log_2 x)}.$$

Hence,

$$\sigma(m) < x^{1/10}$$

for large x.

We will apply Lemma 2 with

$$y = x, \quad K \text{ playing the role of } k \text{ in the lemma}, \quad \tilde{u}_i, \tilde{v}_i \text{ in place of } u_i, v_i,$$
$$B = \sigma(m), \quad d \text{ as before}, \quad l \text{ as before (i.e., } l = L-k \text{ for our original } k).$$

Since all of the previous hypotheses except (possibly) (i) held for our solution to (4), all of the statements in Lemma 2 are satisfied for our solution to (7). That is, with this choice of parameters, Lemma 2 succeeds in capturing our solution to (7).

Now given m, one can recover the original solution to (4) from the solution to the canceled form (7). So to count solutions to (4), it is enough to sum the upper bound of the lemma over possible values of the parameters d, m (which determines $B = \sigma(m)$), k (which determines $l = L-k$), K, the $\tilde{u}_i$, and $\tilde{v}_i$.

3.2.5 Finishing up

Making analogous calculations to those on [7, pp. 1693–1694], the upper bound arising from a single application of Lemma 2 is seen to be

$$\ll \frac{x}{\sigma(m)\log x} \exp(-\frac{1}{3}(\log_2 x)^{1/2+\varepsilon/2}) \frac{L^{\Omega(d)}}{d}.$$

(compare with [7, eq. (5-13)]). It remains to sum on d, k, K, m, $\tilde{u}_i$, and $\tilde{v}_i$.

Since each of k and K is bounded by L, there are only $O((\log_3 x)^2)$ possibilities for the pair (k,K). Reasoning as in [7, eq. (5-14)], the number of possibilities for the $\tilde{u}_i$ and $\tilde{v}_i$ is bounded by $\exp(O((\log_3 x)^2))$.

To handle the sum on d, we first establish the uniform bound

$$\Omega(d) \ll (\log_2 x)^{1/2}.$$

Recall that we defined d so that $d = \sigma(h)$ for some unitary divisor h of a supported on primes $\leq p_L$. If $h \leq \exp((\log_2 x)^{1/2})$, then the desired bound on $\Omega(d)$ follows from the estimate $\Omega(d) \ll \log d$. Otherwise, conditions (3) and (4) in the definition of $\mathscr{A}_\sigma$ give us

$$\Omega(d) \leq 10 \log_2 \sigma(h) \ll \log_2 h \leq \log_2 p_L^{10\log_2 x},$$

and this is also $O((\log_2 x)^{1/2})$, by the calculation in the final display of [7, p. 1694]. It follows that

$$L^{\Omega(d)} \leq \exp(O((\log_2 x)^{1/2} \log_4 x)).$$

Recall that $P^+(d) \leq v_k$. Our choice of v_k now yields $\log_2 P^+(d) \leq (\log_2 x)^{1/2+\varepsilon/5}$ [7, eq. (5-12)]. Hence,

$$\sum \frac{1}{d} \ll \exp((\log_2 x)^{1/2+\varepsilon/5}).$$

Assembling the preceding estimates, we find that the number of solutions to (4) corresponding to a given value of $m = \gcd(p_0 \dots p_{k-1}, q_0 \cdots q_{k-1})$ is at most

$$\frac{x}{\sigma(m)\log x} \exp\left(-(\log_2 x)^{1/2}\right).$$

It remains finally to treat the sum over m. Since m divides $p_1 \cdots p_{k-1}$, we have that m is squarefree and (recalling (5))

$$\omega(m) < k < \frac{1}{2}L < 0.9 \log_3 x.$$

Hence,

$$\sum_m \frac{1}{\sigma(m)} \leq \sum_{j \leq 0.9\log_3 x} \frac{1}{j!} \left(\sum_{p \leq x} \frac{1}{p+1}\right)^j < \exp((\log_3 x)^2),$$

by a short calculation using Mertens's estimate $\sum_{p \leq x} p^{-1} = \log_2 x + O(1)$. We conclude that the total number of solutions to (4) is bounded by (say)

$$\frac{x}{\log x} \exp\left(-\frac{1}{2}(\log_2 x)^{1/2}\right).$$

As discussed before, this is also an upper bound on the number of exceptional σ-values in $[1,x]$ all of whose preimages belong to $\mathscr{A}_\sigma$. Since this upper bound is certainly $o(V_\sigma(x))$, the proof of Theorem 2 is complete.

Remark 5. Recall that $\varepsilon > 0$ can be taken arbitrarily small in the definition of $\mathscr{A}_\sigma$. From the above argument and Proposition 2, it follows that the number of exceptional σ-values in $[1,x]$ is at most $V_\sigma(x)/(\log_2 x)^{1/2+o(1)}$, as $x \to \infty$.

Remark 6. Using the preceding remark and further ideas from [6] and [7], one can establish the following strengthening of Theorem 2: *For each fixed K, almost all σ-values in $[1,x]$ are such that all of their preimages share the same largest $K+1$ prime factors*. One could even extend this to certain functions $K = K(x) \to \infty$, but we do not pursue this possibility here.

Remark 7. Theorem 2 as well as the comments in the preceding remarks all hold with σ replaced by Euler's φ-function, by essentially the same proofs.

4 Proof of Corollary 1

Proof. Suppose that $v \leq x$ is the common σ-value of some amicable tuple. Then there are $n_1, \dots, n_k$ with $\sum_{i=1}^{k} n_i = v$ and each $\sigma(n_i) = v$. Clearly, each $n_i \leq x$. By Theorem 2, we can assume that all the n_i have the same largest prime factor P. We can also assume that $P > z$, where $z := x^{1/(4\log\log x)}$. Otherwise, a crude upper bound on the count of smooth numbers (such as [17, Theorem 1, p. 359]) shows that each n_i is restricted to a set of size $\ll x/(\log x)^2 = o(V_\sigma(x))$, which would mean that $v = \sigma(n_1)$ is also so restricted. For the remaining values of v, observe that P divides $\sum_{i=1}^{k} n_i = v = \sigma(n_1)$. Write $n_1 = Pm$. If $P \mid m$, then n_1 is divisible by the square of a prime exceeding z, leaving $\ll x\sum_{p>z} p^{-2} \ll x/z = o(V_\sigma(x))$ possibilities for n_1. So assume that $P \nmid m$. Since P divides $\sigma(n_1) = (P+1)\sigma(m)$ and $P = P^+(n_1)$, there must be a proper prime power R dividing m for which $P \mid \sigma(R)$. Since P divides $\sigma(R)$ and $\sigma(R) < 2R$, we have that $R > P/2 \geq z/2$. Thus, n_1 possesses a squarefull divisor exceeding $z/2$, which restricts n_1 — and also $v = \sigma(n_1)$ — to a set of size $\ll x/\sqrt{z} = o(V_\sigma(x))$.

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