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# A REMARK ON PRIME DIVISORS OF PARTITION FUNCTIONS

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Schinzel showed that the set of primes that divide some value of the classical partition function is infinite. For a wide class of sets $\mathscr{A}$, we prove an analogous result for the function $p_{\mathscr{A}}(n)$ that counts partitions of $n$ into terms belonging to $\mathscr{A}$.

*Keywords*: restricted partition function; Bateman–Erdős conjecture; Erdős–Ivić conjecture; linear forms in logarithms.

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## 1. Introduction

Let $p(n)$ denote the classical partition function, defined as the number of ways of writing $n$ as a sum of positive integers, where the order of the summands is not taken into account. Hardy and Ramanujan [10] developed the circle method in order to obtain precise estimates of the asymptotic behavior of $p(n)$ as $n \to \infty$. Their results were later refined by Rademacher [13], who found an exact expression for $p(n)$ as the sum of a rapidly converging series. Taking the first term in Rademacher's series results in the stunning asymptotic formula

$$p(n) = \frac{e^{\pi\sqrt{2/3}\sqrt{n-\frac{1}{24}}}}{4\sqrt{3}(n-\frac{1}{24})}\left(1 - \frac{\sqrt{3/2}/\pi}{\sqrt{n-\frac{1}{24}}}\right) + O\left(\exp\left(\frac{1}{2}\pi\sqrt{2/3}\sqrt{n}\right)\right). \quad (1.1)$$

In addition to its asymptotic properties, the arithmetic properties of the partition function $p(n)$ have drawn the attention of a number of authors. Once again, the story begins with Ramanujan, who discovered the remarkable congruences

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11},$$

valid for all nonnegative integers $n$. As a weak consequence of this result, 5, 7, and 11 each divide infinitely many values of $p(n)$. Seventy years later, Schinzel showed that infinitely many primes divide some member of the sequence $\{p(n)\}_{n=0}^{\infty}$. (See [8, Lemma 2.1]; see also [14] for quantitative results.) Schinzel's theorem was superseded by work of Ono [12], who showed that *every* prime divides some value of $p(n)$. Very roughly speaking, Ono uses the theory of modular forms to show that congruences of the sort discovered by Ramanujan are surprisingly ubiquitous. A useful survey of related work is given in [1].

Despite these recent developments, Schinzel's method still has some life to it. Quite recently, Cilleruelo and Luca [7] used a sophisticated variant of Schinzel's argument to prove that the largest prime factor of $p(n)$ exceeds $\log\log n$ for (asymptotically) almost all $n$ (see also the weaker result [11]). The purpose of this note is to establish the analogue of Schinzel's original theorem for a wide class of restricted partition functions.

If $\mathscr{A}$ is a set of positive integers, we write $p_{\mathscr{A}}(n)$ for the number of partitions of $n$ into parts all of which belong to $\mathscr{A}$. Equivalently, $p_{\mathscr{A}}(n)$ is defined by the generating function identity $\sum_{n=0}^{\infty} p_{\mathscr{A}}(n)z^n = \prod_{a\in\mathscr{A}}(1-z^a)^{-1}$.

**Theorem 1.1.** *Let $\mathscr{A}$ be an infinite set of positive integers. Suppose that $\mathscr{A}$ satisfies the following hypothesis, referred to hereafter as* 'condition $P$'*:*

*There is no prime dividing all sufficiently large elements of $\mathscr{A}$.* $(P)$

*Then the set of primes that divide some nonzero value of $p_{\mathscr{A}}(n)$ is infinite.*

For example, Theorem 1.1 applies if $\mathscr{A}$ is taken to be the set of perfect $r$th powers or the set of $r$th powers of primes, for any $r \geq 1$. Asymptotics for these partition functions were studied by Hardy and Ramanujan (see [9, §5]). It seems plausible to conjecture that the conclusion of Theorem 1.1 in fact holds for *every* infinite set $\mathscr{A}$.

Theorem 1.1 does not apply to finite sets $\mathscr{A}$, but for these sets the situation is much simpler. Let $\mathscr{A}$ be a $k$-element set with $k \geq 2$. There is a degree $k-1$ polynomial $f(z) \in \mathbf{Q}[z]$ with the property that $p_{\mathscr{A}}(n\prod_{a\in\mathscr{A}} a) = f(n)$ for every natural number $n$. (See, for example, [2].) For all but finitely many primes $p$, the coefficients of $f$ are $p$-integral, and so $f$ can be reduced mod $p$. An elementary variant of Euclid's proof of the infinitude of primes, dating back at least to Schur [15, pp. 40–41], shows that the reduction of $f$ has a root mod $p$ for infinitely many primes $p$. In fact, the (non-elementary) Frobenius density theorem shows that $f$ has a root mod $p$ for a positive proportion of all primes $p$. Thus, infinitely many primes — in fact, a positive proportion of all primes — divide a nonzero value of $p_{\mathscr{A}}(n)$.

At present, Theorem 1.1 appears to be one of only a handful of theorems in the literature concerning arithmetic properties of the partition functions $p_{\mathscr{A}}(n)$. Other examples, concerned with the parity of $p_{\mathscr{A}}(n)$, can be found in the papers of Berndt, Yee, and Zaharescu [5,6].

## 2. Asymptotic results for $p_{\mathscr{A}}(n)$

On the face of it, Schinzel's argument depends crucially on the existence of the extremely precise asymptotic relation (1.1). In particular, it is important that the relative error there is of size $O_K(n^{-K})$ for each fixed $K$. Except for very special sets

$\mathscr{A}$, this sharp of a result for $p_{\mathscr{A}}(n)$ is unavailable, and so certain auxiliary estimates must be developed in order to get Schinzel's approach off the ground. We collect the needed results in this section. Throughout, all implied constants may depend on the choice of $\mathscr{A}$ without further mention.

We need some notation. Let $\nabla$ denote the backwards-difference operator, defined by $\nabla f(n) = f(n) - f(n-1)$. For each nonnegative integer $k$, we let $\nabla^{(k)}$ denote the $k$th iterate of $\nabla$. We let $p_{\mathscr{A}}^{(k)}(n) = \nabla^{(k)} p_{\mathscr{A}}(n)$. We have the formal identity

$$\sum_{n=0}^{\infty} p_{\mathscr{A}}^{(k)}(n) z^n = (1-z)^k \sum_{n=0}^{\infty} p_{\mathscr{A}}(n) z^n.$$

The following estimates are contained in more general results of Bateman and Erdős (see [3, Theorem 5]).

**Proposition 2.1.** *Let $\mathscr{A}$ be an infinite set of positive integers. Suppose that $\mathscr{A}$ satisfies condition $P$. Then for each fixed nonnegative integer $k$,*

$$p_{\mathscr{A}}^{(k)}(n) \to \infty \quad \text{as} \quad n \to \infty.$$

*Moreover, as $n \to \infty$,*

$$\frac{p_{\mathscr{A}}^{(k+1)}(n)}{p_{\mathscr{A}}^{(k)}(n)} \to 0. \tag{2.1}$$

Bateman and Erdős conjectured [3, p. 12] that (2.1) could be sharpened to

$$\frac{p_{\mathscr{A}}^{(k+1)}(n)}{p_{\mathscr{A}}^{(k)}(n)} \ll_k n^{-1/2} \tag{2.2}$$

for all large enough $n$. In fact, for each fixed $k$, they conjectured that (2.2) holds as long as $\mathscr{A}$ satisfies a condition called $P_k$:

> *There are more than $k$ elements of $\mathscr{A}$, and if we remove an arbitrary subset of $k$ elements from $\mathscr{A}$, then the remaining elements have gcd 1.* $(P_k)$

We are assuming that $\mathscr{A}$ is infinite, and so our condition $P$ guarantees that all of the conditions $P_k$ hold at once. Thus, the Bateman–Erdős conjecture implies that under condition $P$, (2.2) holds for every $k$ and all $n > n_0(k)$.

The conjecture of Bateman and Erdős has since been proved by Bell [4]. Writing $p_{\mathscr{A}}^{(k)}(n) = p_{\mathscr{A}}(n) \prod_{j=0}^{k-1} \left( p_{\mathscr{A}}^{(j+1)}(n) / p_{\mathscr{A}}^{(j)}(n) \right)$, we obtain the following simple consequence of Bell's theorem.

**Proposition 2.2.** *Let $\mathscr{A}$ be an infinite set of positive integers satisfying condition $P$. Then for every fixed nonnegative integer $k$, and all large $n$, we have*

$$p_{\mathscr{A}}^{(k)}(n) \ll_k p_{\mathscr{A}}(n) n^{-k/2}. \tag{2.3}$$

For our proof of Theorem 1.1, it is necessary to have an upper bound on the successive differences of $\log p_{\mathscr{A}}(n)$. We bootstrap our way there, taking (2.3) as our starting point. As an intermediate step, we study the successive differences of $p_{\mathscr{A}}(n)^j$.

For the remainder of this section, we assume that $\mathscr{A}$ is infinite and satisfies condition $P$.

**Lemma 2.3.** *Let $j$ and $k$ be fixed positive integers. For all large $n$,*

$$\nabla^{(k)} p_{\mathscr{A}}(n)^j \ll_{j,k} p_{\mathscr{A}}(n)^j \cdot n^{-k/2}.$$

**Proof.** The proof is by induction, based on the product rule for $\nabla$:

$$\nabla(fg)(n) = f(n-1)\nabla g(n) + g(n)\nabla f(n). \tag{2.4}$$

We call a formal expression in $n$ a *partition monomial* if it is a product of terms of the form $p_{\mathscr{A}}^{(\ell)}(n-m)$, where $\ell$ and $m$ are nonnegative integers. The number of terms in the product will be called the *degree* of the monomial, while the sum of the values of $\ell$ will be called the *weight*. For example,

$$p_{\mathscr{A}}(n-2) \cdot p_{\mathscr{A}}(n-3) \cdot p_{\mathscr{A}}^{(2)}(n) \cdot p_{\mathscr{A}}^{(1)}(n-1)$$

has degree 4 and weight $0+0+2+1=3$. Applying $\nabla$ to a monomial yields a sum of monomials of the same degree and one higher weight. (This follows from (2.4), by induction on the degree.) Applying this $k$ times, $\nabla^{(k)} p_{\mathscr{A}}(n)^j$ can be written as a sum of monomials of degree $j$ and weight $k$.

Now we estimate the size of an arbitrary monomial, viewed as a function of $n$. Fix nonnegative integers $\ell$ and $m$. From (2.3), $p_{\mathscr{A}}^{(\ell)}(n-m) \ll p_{\mathscr{A}}(n-m)n^{-\ell/2}$ for large $n$. From (2.1) (with $k=0$), we have $p_{\mathscr{A}}(n-1) \sim p_{\mathscr{A}}(n)$ as $n \to \infty$, and iterating this shows that $p_{\mathscr{A}}(n-m) \sim p_{\mathscr{A}}(n)$. Hence, $p_{\mathscr{A}}^{(\ell)}(n-m) \ll p_{\mathscr{A}}(n)n^{-\ell/2}$. Multiplying inequalities of this type together, we deduce that a fixed partition monomial of weight $k$ and degree $j$ is $O_{j,k}(p_{\mathscr{A}}(n)^j \cdot n^{-k/2})$ for large $n$. Since $\nabla^{(k)} p_{\mathscr{A}}(n)^j$ is a sum of $O_{j,k}(1)$ such monomials, the lemma follows. □

We can now prove the needed result about the successive differences of $\log p_{\mathscr{A}}(n)$.

**Lemma 2.4.** *Let $k$ be a fixed positive integer. Then for all large $n$, we have*

$$\nabla^{(k)} \log p_{\mathscr{A}}(n) \ll_k n^{-k/2}.$$

**Proof.** We start by writing

$$\nabla^{(k)} \log p_{\mathscr{A}}(n) = \sum_{m=0}^{k} \binom{k}{m} (-1)^m \log p_{\mathscr{A}}(n-m) \tag{2.5}$$

$$= \sum_{m=0}^{k} \binom{k}{m} (-1)^m \log \frac{p_{\mathscr{A}}(n-m)}{p_{\mathscr{A}}(n)}. \tag{2.6}$$

Let $0 \leq m \leq k$. Repeated application of the $k = 1$ case of (2.3) shows that $p_{\mathscr{A}}(n) - p_{\mathscr{A}}(n-m) \ll_k p_{\mathscr{A}}(n) n^{-1/2}$ for large $n$. Consequently,

$$\begin{aligned}\log \frac{p_{\mathscr{A}}(n-m)}{p_{\mathscr{A}}(n)} &= \log\left(1 + \left(\frac{p_{\mathscr{A}}(n-m)}{p_{\mathscr{A}}(n)} - 1\right)\right) \\ &= \sum_{\ell=1}^{k-1} \frac{(-1)^{\ell-1}}{\ell} \left(\frac{p_{\mathscr{A}}(n-m)}{p_{\mathscr{A}}(n)} - 1\right)^{\ell} + O_k(n^{-k/2}).\end{aligned}$$

If we substitute this back into (2.6), the accumulated error term is $O_k(n^{-k/2})$. To estimate the main term, write

$$\begin{aligned}&\sum_{\ell=1}^{k-1} \frac{(-1)^{\ell-1}}{\ell} \left(\frac{p_{\mathscr{A}}(n-m)}{p_{\mathscr{A}}(n)} - 1\right)^{\ell} \\ &\qquad = A_{k-1} \left(\frac{p_{\mathscr{A}}(n-m)}{p_{\mathscr{A}}(n)}\right)^{k-1} + A_{k-2} \left(\frac{p_{\mathscr{A}}(n-m)}{p_{\mathscr{A}}(n)}\right)^{k-2} + \cdots + A_0,\end{aligned}$$

where $A_0, \ldots, A_{k-1}$ are rational numbers depending only on $k$. Using this in (2.6), we obtain a main term of

$$\begin{aligned}&\sum_{m=0}^{k} \binom{k}{m} (-1)^m \sum_{j=0}^{k-1} A_j \left(\frac{p_{\mathscr{A}}(n-m)}{p_{\mathscr{A}}(n)}\right)^{j} \\ &\qquad = \sum_{j=0}^{k-1} A_j p_{\mathscr{A}}(n)^{-j} \sum_{m=0}^{k} \binom{k}{m} (-1)^m p_{\mathscr{A}}(n-m)^j \\ &\qquad = \sum_{j=0}^{k-1} A_j p_{\mathscr{A}}(n)^{-j} \cdot \nabla^{(k)} p_{\mathscr{A}}(n)^j.\end{aligned}$$

The $j = 0$ term contributes nothing to the final sum, while by Lemma 2.3, the terms $j = 1, \ldots, k-1$ each contribute $O_k(n^{-k/2})$. The lemma follows. □

## 3. Proof of Theorem 1.1

The following result, which is a consequence of Baker's estimates for linear forms in logarithms, is due to Tijdeman [16].

**Proposition 3.1.** *Fix a positive integer $K$. There is a constant $C = C(K)$ with the following property: If $N, M > 3$ are integers both supported on the first $K$ primes, with $N > M$, then*

$$N - M > M/(\log M)^C.$$

**Proof of Theorem 1.1.** Proposition 2.1 shows that $p_{\mathscr{A}}(n) \to \infty$ as $n \to \infty$. So we may fix an $n_0$ with the property that $p_{\mathscr{A}}(n) > 0$ once $n > n_0$. Let us suppose for the sake of contradiction that no term of the sequence $\{p_{\mathscr{A}}(n)\}_{n=n_0}^{\infty}$ is divisible by a prime larger than the $K$th prime, where $K$ is a fixed positive integer. Let $C = C(K)$

be as in Theorem C, and let $k$ denote the smallest positive integer exceeding $C$. For the rest of this proof, all implied constants may depend on $\mathscr{A}$, $K$, and $k$.

Observe that there are arbitrarily large $n$ for which $\nabla^{(k)} \log p_{\mathscr{A}}(n) \neq 0$. Otherwise, $\log p_{\mathscr{A}}(n)$ eventually coincides with a polynomial in $n$. Since $p_{\mathscr{A}}(n) \to \infty$ as $n \to \infty$, that polynomial must be nonconstant with a positive leading coefficient. However, from (1.1), $\log p_{\mathscr{A}}(n) \leq \log p(n) \ll \sqrt{n}$. This is a contradiction.

From now on, we assume $n$ is chosen so that $\nabla^{(k)} \log p_{\mathscr{A}}(n) \neq 0$. From the expression (2.5) for $\nabla^{(k)} \log p_{\mathscr{A}}(n)$, we see that

$$\exp(\nabla^{(k)} \log p_{\mathscr{A}}(n)) = \frac{A}{B},$$
$$\text{where} \quad A := \prod_{\substack{0 \leq m \leq k \\ m \text{ even}}} p_{\mathscr{A}}(n-m)^{\binom{k}{m}}, \quad B := \prod_{\substack{0 \leq m \leq k \\ m \text{ odd}}} p_{\mathscr{A}}(n-m)^{\binom{k}{m}}.$$

Let $N = \max\{A, B\}$ and $M = \min\{A, B\}$. Then $N > M$ and, assuming $n$ is large enough, both $N$ and $M$ are supported on the first $K$ primes. By Lemma 2.4, we have $\nabla^{(k)} \log p_{\mathscr{A}}(n) \ll n^{-k/2}$ for large $n$. It follows that both $A/B$ and $B/A$ are of the form $1 + O(n^{-k/2})$, and this shows that

$$N - M \ll N \cdot n^{-k/2}. \tag{3.1}$$

Both $N$ and $M$ are products of $2^{k-1}$ terms from the set $\{p_{\mathscr{A}}(n), \dots, p_{\mathscr{A}}(n-k)\}$. Since $p_{\mathscr{A}}(n) \sim \cdots \sim p_{\mathscr{A}}(n-k)$ as $n \to \infty$, we get that $N \sim M \sim p_{\mathscr{A}}(n)^{2^{k-1}}$. Consequently,

$$n^{1/2} \gg \log p(n) \geq \log p_{\mathscr{A}}(n) \gg \log M.$$

So from (3.1),

$$N - M \ll M/(\log M)^k.$$

But $k > C$. Thus, for $n$ sufficiently large (so that $N$ and $M$ are also large), this contradicts Proposition 3.1. □

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