QUASI-AMICABLE NUMBERS ARE RARE

PAUL POLLACK

ABSTRACT. Define a *quasi-amicable pair* as a pair of distinct natural numbers each of which is the sum of the nontrivial divisors of the other, e.g., $\{48, 75\}$. Here *nontrivial* excludes both 1 and the number itself. Quasi-amicable pairs have been studied (primarily empirically) by Garcia, Beck and Najar, Lal and Forbes, and Hagis and Lord. We prove that the set of n belonging to a quasi-amicable pair has asymptotic density zero.

1. INTRODUCTION

Let $s(n) := \sum_{d|n, d<n} d$ be the sum of the proper divisors of n. Given a natural number n, what can one say about the *aliquot sequence at* n defined as $n, s(n), s(s(n)), \dots$? From ancient times, there has been considerable interest in the case when this sequence is purely periodic. (In this case, n is called a *sociable number*; see Kobayashi et al. [11] for some recent results on such numbers.) An n for which the period is 1 is called *perfect* (see sequence A000396), and an n for which the period is 2 is called *amicable* (see sequence A063990). In the latter case, we call $\{n, s(n)\}$ an *amicable pair*.

Let $s^-(n) := \sum_{d|n, 1<d<n} d$ be the sum of the nontrivial divisors of the natural number n, where *nontrivial* excludes both 1 and n. According to Lal and Forbes [12], it was Chowla who suggested studying *quasi-aliquot sequences* of the form $n, s^-(n), s^-(s^-(n)), \dots$. Call n *quasi-amicable* if the quasi-aliquot sequence starting from n is purely periodic of period 2 (see sequence A005276). Thus, a *quasi-amicable* pair is a pair of distinct natural numbers n and m with $s^-(n) = m$ and $s^-(m) = n$ (e.g., $n = 48$ and $m = 75$). The numerical data, reproduced in Table 1 from sequence A126160, suggests that the number of such pairs with a member $\le N$ tends to infinity with N, albeit very slowly.

While quasi-amicable pairs have been studied empirically (see [8, 12, 1, 10, 2], and cf. [14, 13], [9, section B5]), it appears that very little theoretical work has been done. In this paper, we prove the following modest theorem, which is a quasi-amicable analogue of Erdős's 1955 result [4] concerning amicable pairs:

Theorem 1.1. *The set of quasi-amicable numbers has asymptotic density zero. In fact, as* $\epsilon \downarrow 0$*, the upper density of the set of* n *satisfying*

$$1 - \epsilon < \frac{s^-(s^-(n))}{n} < 1 + \epsilon \tag{1.1}$$

tends to zero.

Remark. With s replacing s^-, Theorem 1.1 follows from work of Erdős [4] and Erdős et al. [7, Theorem 5.1].

2010 *Mathematics Subject Classification.* Primary: 11A25, Secondary: 11N37.

Key words and phrases. aliqot sequence, quasi-aliquot sequence, quasi-amicable pair, augmented amicable pair.

N	# of quasi-amicable pairs with least member $\leq N$
10^5	9
10^6	17
10^7	46
10^8	79
10^9	180
10^{10}	404
10^{11}	882
10^{12}	1946

TABLE 1

Notation. Throughout, p and q always denote prime numbers. We use $\sigma(n) := \sum_{d|n} d$ for the sum of all positive divisors of n, and we let $\omega(n) := \sum_{p|n} 1$ stand for the number of distinct prime factors of n. We write $P(n)$ for the largest prime divisor of n, with the understanding that $P(1) = 1$. We say that n is *y-smooth* if $P(n) \leq y$. For each n, its *y-smooth part* is defined as the largest y-smooth divisor of n.

The Landau–Bachmann o and O-symbols, as well as Vinogradov's $\ll$ notation, are employed with their usual meanings. *Implied constants are absolute unless otherwise specified.*

2. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1, assuming two preliminary results whose proofs are deferred to §3 and §4.

Proposition 2.1. *For each $\epsilon > 0$, the set of natural numbers n with*

$$\frac{\sigma(n+1)}{n+1} - \epsilon < \frac{\sigma(s^-(n))}{s^-(n)} < \frac{\sigma(n+1)}{n+1} + \epsilon. \tag{2.1}$$

has asymptotic density 1.

Remark. If n is prime, then $s^-(n) = 0$, and the expression $\sigma(s^-(n))/s^-(n)$ is undefined. This does not contradict Proposition 2.1, since the set of primes has asymptotic density zero.

Proposition 2.2. *As $\epsilon \downarrow 0$, the upper density of the set of natural numbers n for which*

$$1 - \epsilon < \left(\frac{\sigma(n)}{n} - 1\right)\left(\frac{\sigma(n+1)}{n+1} - 1\right) < 1 + \epsilon \tag{2.2}$$

tends to zero.

Proof of Theorem 1.1. It suffices to prove the upper density assertion of the theorem. Let $\delta > 0$. We will show that if $\epsilon > 0$ is sufficiently small, then the upper density of the set of n for which (1.1) holds is at most 2δ. We start by assuming that both $\sigma(n)/n \leq B$ and $\sigma(n+1)/(n+1) \leq B$, where $B > 0$ is chosen so that these conditions exclude a set of n of upper density at most δ. To see that such a choice is possible, we can use a first moment argument; indeed, since

$$\sum_{n\leq x} \frac{\sigma(n)}{n} = \sum_{n\leq x}\sum_{d|n} \frac{1}{d} \leq x \sum_{d\leq x} \frac{1}{d^2} < 2x,$$

we can take $B = 4/\delta$. Moreover, Proposition 2.1 shows that by excluding an additional set of density 0, we can assume that

$$\left|\frac{\sigma(s^-(n))}{s^-(n)} - \frac{\sigma(n+1)}{n+1}\right| < \frac{\epsilon}{2B}.$$

Now write

$$\begin{aligned}\frac{s^-(s^-(n))}{n} &= \frac{s^-(n)}{n}\frac{s^-(s^-(n))}{s^-(n)} \\ &= \left(\frac{\sigma(n)}{n} - 1 - \frac{1}{n}\right)\left(\frac{\sigma(s^-(n))}{s^-(n)} - 1 - \frac{1}{s^-(n)}\right).\end{aligned}$$

If n is a large natural number satisfying (1.1) and our above conditions, then a short computation shows $\frac{s^-(s^-(n))}{n}$ is within ϵ of the product $(\frac{\sigma(n)}{n} - 1)(\frac{\sigma(n+1)}{n+1} - 1)$. (Keep in mind that since n is composite, we have $s^-(n) \geq \sqrt{n}$.) Thus,

$$1 - 2\epsilon < \left(\frac{\sigma(n)}{n} - 1\right)\left(\frac{\sigma(n+1)}{n+1} - 1\right) < 1 + 2\epsilon.$$

Finally, Proposition (2.2) shows that if ϵ is chosen sufficiently small, then these remaining n make up a set of upper density $< \delta$. □

3. The proof of Proposition 2.1

3.1. **Preparation.** The proof of the proposition is very similar to the proof, due to Erdős, Granville, Pomerance, and Spiro, that $s(s(n))/s(n) = s(n)/n + o(1)$, as $n \to \infty$ along a sequence of density 1 (see Erdős et al. [7, p. 195]). We follow their argument, as well as the author's adaptation [15], very closely.

We begin by recalling some auxiliary estimates. The first of these is due to Pomerance [16, Theorem 2].

Lemma 3.1. *Let D be a natural number, and let $x \geq 2$. The number of $n \leq x$ for which $D \nmid \sigma(n)$ is $\ll x/(\log x)^{1/\varphi(D)}$.*

For a given α, we call the natural number n an *α-primitive* number if $\sigma(n)/n \geq 1 + \alpha$ while $\sigma(d)/d < 1 + \alpha$ for every proper divisor d of n. The following estimate is due to Erdős [5, p. 6]:

Lemma 3.2. *Fix a positive rational number α. There is a constant $c = c(\alpha) > 0$ and an $x_0 = x_0(\alpha)$ so that for $x > x_0$, the number of α-primitive $n \leq x$ is at most*

$$\frac{x}{\exp(c\sqrt{\log x \log\log x})}.$$

As a consequence of Lemma 3.2, we obtain the following convergence result, which we will need to conclude the proof of Proposition 2.1.

Lemma 3.3. *Fix a positive rational number α. Then*

$$\sum_{a\ \alpha\text{-primitive}} \frac{2^{\omega(a)}}{a} < \infty.$$

Proof. We split the values of a appearing in the sum into two classes, putting those a for which $\omega(a) \leq 20 \log\log a$ in the first class and all other a in the second. If a belongs to the first class, then $2^{\omega(a)} \leq (\log a)^{20 \log 2}$, and Lemma 3.2 shows that the sum over these a converges (by partial summation). To handle the a in the second class, we ignore the α-primitivity condition altogether and invoke a lemma of Pollack [15, Lemma 2.4], according to which $\sum_{a:\ \omega(a) > 20 \log\log a} \frac{2^{\omega(a)}}{a} < \infty$. □

3.2. **Proof proper.** We proceed to prove Proposition 2.1 in two stages; first we prove that the lower-bound holds almost always, and then we do the same for the upper bound. The following lemma is needed for both parts.

Lemma 3.4. *Fix a natural number T. For each composite value of n with $1 \leq n \leq x$, write*

$$n + 1 = m_1 m_2 \quad \textit{and} \quad s^-(n) = M_1 M_2,$$

where $P(m_1 M_1) \leq T$ and every prime dividing $m_2 M_2$ exceeds T. Then, except for $o(x)$ (as $x \to \infty$) choices of n, we have $m_1 = M_1$.

Proof. At the cost of excluding $o(x)$ values of $n \leq x$, we may assume that

$$m_1 \leq (\log\log x)^{1/2} \left(\prod_{p \leq T} p\right)^{-1} =: R.$$

Indeed, in the opposite case, $n + 1$ has a T-smooth divisor exceeding R, and the number of such $n \leq x$ is

$$\ll x \sum_{\substack{e\ T\text{-smooth} \\ e > R}} \frac{1}{e} = o(x),$$

as $x \to \infty$. Here we use that the sum of the reciprocals of the T-smooth numbers is $\prod_{p \leq T}(1 - 1/p)^{-1} < \infty$. Hence, $m_1 \prod_{p \leq T} p \leq (\log\log x)^{1/2}$, and so Lemma 3.1 shows that excluding $o(x)$ values of $n \leq x$, we can assume that $m_1 \prod_{p \leq T} p$ divides $\sigma(n)$. Since

$$s^-(n) = \sigma(n) - (n + 1),$$

it follows that m_1 is the T-smooth part of $s^-(n)$. That is, $m_1 = M_1$. □

Proof of the lower bound half of Proposition 2.1. Fix $\delta > 0$. We will show that the number of $n \leq x$ for which the left-hand inequality in (2.1) fails is smaller than $3\delta x$, once x is large.

Fix B large enough that $\sigma(n+1)/(n+1) \leq B$ except for at most δx exceptional $n \leq x$. That this is possible follows from the first moment argument used in the proof of Theorem 1.1 (e.g., we may take $B = 4/\delta$ again). Next, fix T large enough so that with m_2 defined as in Lemma 3.4, we have

$$\frac{\sigma(m_2)}{m_2} \leq \exp(\epsilon/B)$$

except for at most δx exceptional $n \le x$. To see that a suitable choice of T exists, observe that

$$\sum_{n\le x} \log\frac{\sigma(m_2)}{m_2} \le \sum_{n\le x}\sum_{\substack{p\mid n+1\\ p>T}} \log\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right)$$
$$\le \sum_{n\le x}\sum_{\substack{p\mid n+1\\ p>T}} \frac{1}{p-1} \le 2x\sum_{p>T}\frac{1}{p(p-1)} < \frac{2x}{T}.$$

Hence, we may take $T = \lceil 2B/(\delta\epsilon)\rceil$.

For large x, we have that n is composite (so that M_1 is defined) and that $m_1 = M_1$, except for at most δx values of $n \le x$. This follows from Lemma 3.4 and the fact that the primes have density 0.

If n is not in any of the exceptional classes defined above, then

$$\frac{\sigma(s^-(n))}{s^-(n)} = \frac{\sigma(M_1M_2)}{M_1M_2} \ge \frac{\sigma(M_1)}{M_1} = \frac{\sigma(m_1)}{m_1} = \frac{\sigma(n+1)/(n+1)}{\sigma(m_2)/m_2}$$
$$\ge \frac{\sigma(n+1)}{n+1}\exp\left(-\frac{\epsilon}{B}\right) > \frac{\sigma(n+1)}{n+1}\left(1-\frac{\epsilon}{B}\right) \ge \frac{\sigma(n+1)}{n+1} - \epsilon,$$

which is the desired lower bound. Note that at most $3\delta x$ values of $n \le x$ are exceptional, as claimed. □

Proof of the upper bound half of Proposition 2.1. We may suppose that $0 < \epsilon < 1$. Let $\delta > 0$ be given. Fix $\eta \in (0,1)$ so small that the number of $n \le x$ which are either prime or which fail to satisfy

$$P(n) > x^{\eta} \quad \text{and} \quad P(n)^2 \nmid n \tag{3.1}$$

is smaller than δx, once x is large. The existence of such an η follows either from Brun's sieve or well-known work of Dickman on smooth numbers. Next, using the first moment argument from the proof of Theorem 1.1, choose a fixed number $B \ge 1$ so that all but at most δx of the numbers $n \le x$ satisfy

$$\frac{\sigma(n+1)}{n+1} \le B. \tag{3.2}$$

We fix rational numbers α_1 and α_2 satisfying

$$0 < \alpha_1 \le \frac{\epsilon}{4B}, \quad 0 < \alpha_2 \le \frac{\alpha_1\eta}{12}.$$

Finally, we fix a natural number T which is sufficiently large, depending only on the α_i, δ, η, and B. The precise meaning of "sufficiently large" will be specified in the course of the proof.

Suppose that the right-hand inequality (2.1) fails for n, where we assume that n is composite and satisfies both (3.1) and (3.2). Write

$$n+1 = m_1m_2 \quad \text{and} \quad s^-(n) = M_1M_2,$$

where $P(m_1M_1) \le T$ and every prime dividing m_2M_2 exceeds T. By Lemma 3.4, we can assume $m_1 = M_1$, excluding at most δx values of $n \le x$. Thus,

$$\frac{\sigma(M_2)/M_2}{\sigma(m_2)/m_2} = \frac{\sigma(s^-(n))/s^-(n)}{\sigma(n+1)/(n+1)} \ge 1 + \frac{\epsilon}{\sigma(n+1)/(n+1)} \ge 1 + \frac{\epsilon}{B} \ge 1 + 4\alpha_1.$$

In particular,

$$\frac{\sigma(M_2)}{M_2} \geq 1 + 4\alpha_1. \tag{3.3}$$

We can assume our choice of T was such that, apart from at most δx exceptional $n \leq x$, we have

$$\frac{\sigma(m_2)}{m_2} \leq 1 + \alpha_1. \tag{3.4}$$

Indeed, the argument for the analogous claim in the proof of the lower-bound shows it is sufficient that $T > 2(\delta \log(1+\alpha_1))^{-1}$. Henceforth, we assume (3.4). Now write $M_2 = M_3 M_4$, where every prime dividing M_3 divides $n+1$, while M_4 is coprime to $n+1$. Note that every prime dividing M_3 divides m_2. Hence,

$$\begin{aligned}\frac{\sigma(M_3)}{M_3} \leq \prod_{p \mid M_3}\left(1 + \frac{1}{p-1}\right) &= \left(\prod_{p \mid M_3} \frac{p^2}{p^2-1}\right) \prod_{q \mid M_3} \frac{q+1}{q} \\ &\leq \left(\prod_{p > T} \frac{p^2}{p^2-1}\right) \frac{\sigma(m_2)}{m_2} \leq 1 + 2\alpha_1,\end{aligned}$$

using (3.4) and assuming an initial appropriate choice of T. So from (3.3),

$$\frac{\sigma(M_4)}{M_4} = \frac{\sigma(M_2)/M_2}{\sigma(M_3)/M_3} \geq \frac{1+4\alpha_1}{1+2\alpha_1} \geq 1 + \alpha_1.$$

It follows that there is an α_1-primitive number a_1 dividing M_4, where each prime dividing a_1 exceeds T.

We claim next that there is a squarefree, α_2-primitive number a_2 dividing a_1 with

$$a_2 \leq a_1^{\eta/2}.$$

List the distinct prime factors of a_1 in increasing order, say $T < q_1 < q_2 < \cdots < q_t$, and put $a_0 := q_1 q_2 \cdots q_{\lfloor \eta t/2 \rfloor}$, so that

$$a_0 \leq (q_1 \cdots q_t)^{\lfloor \eta t/2 \rfloor / t} \leq a_1^{\eta/2}.$$

We will show that $\sigma(a_0)/a_0 \geq 1 + \alpha_2$; then we can take a_2 as any α_2-primitive divisor of a_0. First, observe that $\lfloor \eta t/2 \rfloor \geq \eta t/3$. Otherwise, $t < 6/\eta$ and

$$1 + \alpha_1 \leq \frac{\sigma(a_1)}{a_1} \leq \prod_{1 \leq i \leq t}\left(1 + \frac{1}{q_i - 1}\right) \leq \left(1 + \frac{1}{T}\right)^{6/\eta} \leq \exp\left(\frac{6}{\eta T}\right),$$

which is false, assuming a suitable initial choice of T. It follows that

$$\frac{\sigma(a_0)}{a_0} = \prod_{1 \leq i \leq \lfloor \eta t/2 \rfloor} \frac{q_i + 1}{q_i} \geq \left(\prod_{p > T} \frac{p^2-1}{p^2}\right) \prod_{1 \leq i \leq \lfloor \eta t/2 \rfloor} \frac{q_i}{q_i - 1},$$

while

$$\begin{aligned}\prod_{1 \leq i \leq \lfloor \eta t/2 \rfloor} \frac{q_i}{q_i - 1} &\geq \left(\prod_{1 \leq i \leq t} \frac{q_i}{q_i - 1}\right)^{\lfloor \eta t/2 \rfloor / t} \\ &\geq \left(\frac{\sigma(a_1)}{a_1}\right)^{\eta/3} \geq (1 + \alpha_1)^{\eta/3} \geq 1 + \frac{\alpha_1 \eta}{6}.\end{aligned}$$

Thus,

$$\frac{\sigma(a_0)}{a_0} \geq \left(\prod_{p>T} \frac{p^2-1}{p^2}\right)\left(1+\frac{\alpha_1\eta}{6}\right) \geq 1+\frac{\alpha_1\eta}{12} \geq 1+\alpha_2,$$

again assuming a suitable choice of T to justify the middle inequality.

Observe that a_2 satisfies

$$a_2 \leq a_1^{\eta/2} \leq (s^-(n))^{\eta/2} < x^{2\eta/3},$$

for large x. Write $n = Pr$, where $P = P(n)$. Then $r > 1$ (since n is composite) and also, by (3.1),

$$r \leq x/P \leq x^{1-\eta}.$$

Moreover, a_2 divides

$$s^-(Pr) = P(\sigma(r) - r) + \sigma(r) - 1,$$

and so

$$P(\sigma(r) - r) \equiv 1 - \sigma(r) \pmod{a_2}.$$

We view this as a linear congruence condition on P modulo a_2. If there are any solutions, then $D := \gcd(\sigma(r) - r, a_2) \mid 1 - \sigma(r)$, and in this case there are exactly D solutions modulo a_2. Note that if there are any solutions, then $D \mid r - 1$. Also note that D is squarefree, since a_2 is squarefree.

We now sum over pairs a_2 and r, for each pair counting the number of possible values of $P \leq x/r$. By the Brun–Titchmarsh inequality and the preceding remarks about D, we have that the number of possible values of $n = Pr$ is

$$\ll \sum_{\substack{a_2 \ \alpha_2\text{-primitive} \\ T<a_2\leq x^{2\eta/3}}} \sum_{1<r\leq x^{1-\eta}} \sum_{\substack{D\mid(a_2,r-1) \\ D \text{ squarefree}}} D\frac{x/r}{\varphi(a_2)\log(x/(a_2 r))}$$
$$\ll \frac{x}{\eta\log x} \sum_{\substack{a_2 \ \alpha_2\text{-primitive} \\ T<a_2\leq x^{\eta/3}}} \frac{1}{\varphi(a_2)} \sum_{\substack{D\mid a_2 \\ D \text{ squarefree}}} D \sum_{\substack{1<r\leq x^{1-\eta} \\ D\mid r-1}} \frac{1}{r}.$$

The sum on r is $\ll \frac{1}{D}\log x$. Moreover, since a_2 is α_2-primitive, we have

$$\frac{a_2}{\varphi(a_2)} \ll \frac{\sigma(a_2)}{a_2} \leq \frac{3}{2}(1+\alpha_2) \ll 1,$$

and so $\varphi(a_2) \gg a_2$. Thus, the remaining sum is

$$\ll \frac{x}{\eta} \sum_{\substack{a_2 \ \alpha_2\text{-primitive} \\ T<a_2\leq x^{2\eta/3}}} \frac{1}{a_2} \sum_{\substack{D\mid a_2 \\ D \text{ squarefree}}} 1 \ll \frac{x}{\eta} \sum_{\substack{a_2 \ \alpha_2\text{-primitive} \\ a_2\geq T}} \frac{2^{\omega(a_2)}}{a_2}.$$

But if T was chosen sufficiently large, then this last sum is bounded by $\eta\delta x$ (by Lemma 3.3), leading to an upper bound of $\ll \delta x$. Since the number of exceptional n appearing earlier in the argument is also $\ll \delta x$, and $\delta > 0$ was arbitrary, the proof is complete. □

4. Proof of Proposition 2.2

We start by quoting two lemmas. The first was developed by Erdős [3] to estimate the decay of the distribution function of $\sigma(n)/n$ near infinity. We state the lemma in a slightly stronger form which is supported by his proof.

Lemma 4.1. *For $x > 0$, the number of positive integers $n \leq x$ with $\sigma(n)/n > y$ is*

$$\leq x/\exp(\exp((e^{-\gamma} + o(1))y)), \quad \text{as } y \to \infty,$$

uniformly in x, where γ is the Euler–Mascheroni constant.

The next lemma, also due to Erdős [6], supplies an estimate for how often $\sigma(n)/n$ lands in a short interval; note the uniformity in the parameter a.

Lemma 4.2. *Let $x > t \geq 2$ and let $a \in \mathbb{R}$. The number of $n \leq x$ with $a < \sigma(n)/n < a + 1/t$ is $\ll x/\log t$.*

The next two lemmas develop the philosophy that the rough size of $\sigma(n)/n$ is usually determined by the small prime factors of n. Put $h(n) := \sum_{d|n} \frac{1}{d}$, so that $h(n) = \sigma(n)/n$. For each natural number T, set $h_T(n) := \sum_{d|n, P(d) \leq T} \frac{1}{d}$. The next lemma says that h and h_T are usually close once T is large.

Lemma 4.3. *Let $\epsilon > 0$ and $x \geq 1$. The number of $n \leq x$ with $h(n) - h_T(n) > \epsilon$ is $\ll x/(T\epsilon)$.*

Proof. Again, we use a first moment argument. We have

$$\sum_{n \leq x} (h(n) - h_T(n)) \leq \sum_{n \leq x} \sum_{\substack{d|n \\ d>T}} \frac{1}{d} \leq x \sum_{d>T} \frac{1}{d^2} \ll x/T,$$

from which the result is immediate. □

Lemma 4.4. *Let T be a natural number. Let S be any set of real numbers, and define $\mathscr{E}(S)$ as the set of T-smooth numbers e for which $h_T(e) - 1 \in S$. Then for $n \in \mathbb{N}$, we have $h_T(n) - 1 \in S$ precisely when n has T-smooth part e for some $e \in \mathscr{E}(S)$. Moreover, the density of such n exists and is given by*

$$\sum_{e \in \mathscr{E}(S)} \frac{1}{e} \prod_{p \leq T} (1 - 1/p). \tag{4.1}$$

Proof. It is clear that $h_T(n)$ depends only on the T-smooth part of n. So it suffices to prove that the density of n with T-smooth part in $\mathscr{E}(S)$ is given by (4.1).

For each set of T-smooth numbers $\mathscr{E}$, let $\overline{d}_{\mathscr{E}}$ and $\underline{d}_{\mathscr{E}}$ denote the upper and lower densities of the set of n whose T-smooth part belongs to $\mathscr{E}$. If $\overline{d}_{\mathscr{E}} = \underline{d}_{\mathscr{E}}$, then the density of this set exists; denote it by $d_{\mathscr{E}}$.

For each T-smooth number e, a natural number n has T-smooth part e precisely when e divides n and n/e is coprime to $\prod_{p \leq T} p$, so that the set of such n has density $\frac{1}{e} \prod_{p \leq T}(1 - 1/p)$. Since density is finitely additive, it follows that for any finite subset $\mathscr{E} \subset \mathscr{E}(S)$,

$$d_{\mathscr{E}} = \sum_{e \in \mathscr{E}} \frac{1}{e} \prod_{p \leq T} (1 - 1/p).$$

Now let $x > 0$, and put $\mathscr{E}(S) = \mathscr{E}_1 \cup \mathscr{E}_2$, where $\mathscr{E}_1 = \mathscr{E}(S) \cap [1, x]$ and $\mathscr{E}_2 = \mathscr{E}(S) \setminus \mathscr{E}_1$. Then $\underline{d}_{\mathscr{E}(S)} \geq \underline{d}_{\mathscr{E}_1}$ for all x, and so letting $x \to \infty$, we find that $\underline{d}_{\mathscr{E}(S)}$ is bounded below by

(4.1). On the other hand, $\overline{d}_{\mathscr{E}(S)} \le \overline{d}_{\mathscr{E}_1} + \overline{d}_{\mathscr{E}_2}$. But $\overline{d}_{\mathscr{E}_1}$ is bounded above by (4.1) for all x, while $\overline{d}_{\mathscr{E}_2} \le \sum_{\substack{e\ T\text{-smooth}\\ e>x}} e^{-1} = o(1)$, as $x \to \infty$. Thus, letting $x \to \infty$, we obtain that $\overline{d}_{\mathscr{E}(S)}$ is bounded above by (4.1). ◻

Proof of Proposition 2.2. Let $\delta > 0$ be sufficiently small. We will show that for

$$\epsilon < \exp(-4/\delta), \tag{4.2}$$

the number of $n \le x$ satisfying (2.2) is $\ll \delta(\log\log\frac{1}{\delta})x$, for large x. Note that since $\delta\log\log\frac{1}{\delta} \to 0$ as $\delta \downarrow 0$, this proves the proposition. In what follows, we fix δ and ϵ, always assuming that δ is small and that $\epsilon > 0$ satisfies (4.2).

Put $T := \epsilon^{-1}\delta^{-1}$. We can assume that both n and $n+1$ have T-smooth part $\le \log x$. Indeed, for large x, this excludes a set of n size $< \delta x$, since

$$\sum_{\substack{e\ T\text{-smooth}\\ e>\log x}} \frac{1}{e} = o(1),$$

as $x \to \infty$.

Let I be the closed interval defined by $I := \left[\exp(-1/\delta), 2\log\log\frac{1}{\delta}\right]$. For large x, Lemmas 4.1 and 4.2 imply that all but $\ll \delta x$ values of $n \le x$ are such that $h(n) - 1 \in I$ and $h(n+1) - 1 \in I$. By Lemma 4.3, excluding $\ll \delta x$ additional values of $n \le x$, we can assume that $h_T(n) \ge h(n) - \epsilon$ and $h_T(n+1) \ge h(n+1) - \epsilon$. Recalling the upper bound (4.2) on ϵ, we see that both $h_T(n) - 1$ and $h_T(n+1) - 1$ belong to the interval J, where

$$J := \left[\frac{1}{2}\exp(-1/\delta), 2\log\log\frac{1}{\delta}\right].$$

Moreover (always assuming δ sufficiently small),

$$(h_T(n) - 1)(h_T(n+1) - 1) \ge ((h(n) - 1) - \epsilon)((h(n+1) - 1) - \epsilon) \ge 1 - 5\epsilon\log\log\frac{1}{\delta}, \tag{4.3}$$

and

$$(h_T(n) - 1)(h_T(n+1) - 1) \le (h(n) - 1)(h(n+1) - 1) \le 1 + \epsilon. \tag{4.4}$$

Write J as the disjoint union of $N := \lceil 1/\epsilon \rceil$ consecutive intervals $J_0, J_1, \dots, J_{N-1}$, each of length $1/N$. We estimate, for each $0 \le i < N$, the number of n for which $h_T(n) - 1$ belongs to J_i. Fix $0 \le i < N$. Since $h_T(n) - 1$ belongs to J_i, (4.3) and (4.4) show that

$$h_T(n+1) - 1 \in \left[\frac{1 - 5\epsilon\log\log\frac{1}{\delta}}{x_{i+1}}, \frac{1+\epsilon}{x_i}\right] =: J_i', \tag{4.5}$$

where x_i and x_{i+1} are the left and right endpoints of J_i, respectively. So in the notation of Lemma 4.4, n has T-smooth part $e \in \mathscr{E}(J_i)$ and $n+1$ has T-smooth part $e' \in \mathscr{E}(J_i')$. Clearly, $\gcd(e, e') = 1$. That n and $n+1$ have T-smooth parts e and e', respectively, amounts to a congruence condition on n modulo $M := ee'\prod_{p\le T} p$, where the number of allowable residue classes is $\prod_{p\mid ee'}(p-1)\prod_{p\nmid ee', p\le T}(p-2)$. For large x,

$$M \le (\log x)^2 \prod_{p\le T} p < (\log x)^3 \le x.$$

(Recall that $e, e' \le \log x$.) Thus, the Chinese remainder theorem shows that the number of such $n \le x$ is

$$\ll \frac{x}{ee'} \prod_{p \mid ee'} (1-1/p) \prod_{\substack{p \nmid ee' \\ p \le T}} (1-2/p)$$
$$\le x \left(\frac{1}{e} \prod_{p \le T} (1-1/p)\right) \left(\frac{1}{e'} \prod_{p\le T} (1-1/p)\right) \prod_{p \mid ee'} (1-1/p)^{-1}.$$

But

$$\prod_{p \mid ee'} (1-1/p)^{-1} = \frac{e}{\varphi(e)} \frac{e'}{\varphi(e')} \ll \frac{\sigma(e)}{e} \frac{\sigma(e')}{e'} \ll \left(\log\log \frac{1}{\delta}\right)^2,$$

since $h(e) - 1, h(e') - 1 \le 2 \log\log \frac{1}{\delta}$. Summing over $e \in \mathscr{E}(J_i)$ and $e' \in \mathscr{E}(J_i')$, we find that the number of n under consideration is

$$\ll x \left(\log\log\frac{1}{\delta}\right)^2 \left(\sum_{e \in \mathscr{E}(J_i)} \frac{1}{e} \prod_{p\le T} (1-1/p)\right) \left(\sum_{e' \in \mathscr{E}(J_i')} \frac{1}{e'} \prod_{p \le T}(1-1/p)\right).$$

Now sum over $0 \le i < N$. We obtain that the number of remaining n satisfying (2.2) is $\ll Lx(\log\log\frac{1}{\delta})^2$, where

$$L := \sup_{0 \le i < N} \left\{\sum_{e' \in \mathscr{E}(J_i')} \frac{1}{e'} \prod_{p \le T}(1-1/p)\right\} \left(\sum_{0 \le i < N} \left\{\sum_{e \in \mathscr{E}(J_i)} \frac{1}{e} \prod_{p\le T}(1-1/p)\right\}\right)$$
$$\le \sup_{0 \le i < N} \left\{\sum_{e' \in \mathscr{E}(J_i')} \frac{1}{e'} \prod_{p \le T}(1-1/p)\right\};$$

we use here that the J_i are disjoint, so that

$$\sum_{0 \le i < N} \sum_{e \in \mathscr{E}(J_i)} \frac{1}{e} \le \sum_{e\ T\text{-smooth}} \frac{1}{e} = \prod_{p \le T} (1-1/p)^{-1}.$$

The proof will be completed by showing that $L \ll \delta$. It is enough to argue that each sum

$$\sum_{e' \in \mathscr{E}(J_i')} \frac{1}{e'} \prod_{p \le T} (1-1/p)$$

is $\ll \delta$, uniformly for $0 \le i < N$. By Lemma 4.4, this sum describes the density of those natural numbers m for which $h_T(m) - 1 \in J_i'$. We split these m into two classes, according to whether $h(m) - h_T(m) > \epsilon$ or not. The set of m in the former class has upper density $\ll \delta$, by Lemma 4.3. Suppose now that $h(m)$ belongs to the second class. From the expression (4.5) defining J_i' and a short computation, we see that $h_T(m)$ is trapped within a specific interval of length

$$\ll \exp(2/\delta) \left(\log\log\frac{1}{\delta}\right)^2 \epsilon \ll \exp(3/\delta)\epsilon.$$

Since m belongs to the second class, $h(m)$ is also trapped within a specific interval of length $\ll \exp(3/\delta)\epsilon$. By (4.2), $\exp(3/\delta)\epsilon \leq \exp(-1/\delta)$, and so by Lemma 4.2, the upper density of the set of those m in the second class is

$$\ll \frac{1}{\delta^{-1} + O(1)} \ll \delta,$$

assuming again that δ is sufficiently small. □

Remark. Our argument also shows that the set of *augmented amicable numbers* has density zero (see sequences A007992, A015630). Here an augmented amicable number is an integer which generates a 2-cycle under iteration of the function $s^+(n) := 1 + \sum_{d|n,\ d<n} d$, e.g., $n = 6160$.

Sequences discussed. A000396, A005276, A007992, A015630, A063990, A126160

5. Acknowledgements

The author thanks the National Science Foundation for their generous support (under award DMS-0802970).

References

[1] W. E. Beck and R. M. Najar, More reduced amicable pairs, *Fibonacci Quart.* **15** (1977), 331–332.
[2] ______, Reduced and augmented amicable pairs to 10^8, *Fibonacci Quart.* **31** (1993), 295–298.
[3] P. Erdős, Some remarks about additive and multiplicative functions, *Bull. Amer. Math. Soc.* **52** (1946), 527–537.
[4] ______, On amicable numbers, *Publ. Math. Debrecen* **4** (1955), 108–111.
[5] ______, Remarks on number theory. I. On primitive α-abundant numbers, *Acta Arith.* **5** (1958), 25–33 (1959).
[6] ______, On the distribution of numbers of the form $\sigma(n)/n$ and on some related questions, *Pacific J. Math.* **52** (1974), 59–65.
[7] P. Erdős, A. Granville, C. Pomerance, and C. Spiro, On the normal behavior of the iterates of some arithmetic functions, *Analytic Number Theory (Allerton Park, IL, 1989)*, Progr. Math., vol. 85, Birkhäuser, Boston, 1990, pp. 165–204.
[8] M. Garcia, Números casi amigos y casi sociables, *Revista Annal* (1968).
[9] R. K. Guy, *Unsolved problems in number theory*, third ed., Problem Books in Mathematics, Springer-Verlag, New York, 2004.
[10] P. Hagis, Jr. and G. Lord, Quasi-amicable numbers, *Math. Comp.* **31** (1977), 608–611.
[11] M. Kobayashi, P. Pollack, and C. Pomerance, On the distribution of sociable numbers, *J. Number Theory* **129** (2009), 1990–2009.
[12] M. Lal and A. Forbes, A note on Chowla's function, *Math. Comp.* **25** (1971), 923–925.
[13] H. Mishima, Table of quasi-amicable pairs under 10^{10}, online at http://www.asahi-net.or.jp/~kc2h-msm/mathland/math09/math09t2.htm.
[14] J.M. Pedersen, Tables of aliquot cycles, online at http://amicable.homepage.dk/tables.htm.
[15] P. Pollack, Powerful amicable numbers, *Colloq. Math.* **122** (2011), 103–123.
[16] C. Pomerance, On the distribution of amicable numbers, *J. Reine Angew. Math.* **293/294** (1977), 217–222.

University of Illinois at Urbana-Champaign, Department of Mathematics, 1409 West Green Street, Urbana, Illinois 61801, USA
E-mail address: pppollac@illinois.edu