Multiplicative partitions of numbers with a large squarefree divisor

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Abstract. For each positive integer n, let $f(n)$ denote the number of *multiplicative partitions* of n, meaning the number of ways of writing n as a product of integers larger than 1, where the order of the factors is not taken into account. It was shown by Oppenheim in 1926 that, as $x \to \infty$,

$$\max_{\substack{n \le x \\ n \text{ squarefree}}} f(n) = x/L(x)^{2+o(1)},$$

where $L(x) = \exp(\log x \cdot \frac{\log\log\log x}{\log\log x})$. Without the restriction to squarefree n, the maximum is the significantly larger quantity $x/L(x)^{1+o(1)}$; this was proved by Canfield, Erdős, and Pomerance in 1983. We prove the following theorem that interpolates between these two results: For each fixed $\alpha \in [0, 1]$,

$$\max_{\substack{n \le x \\ \mathrm{rad}(n) \ge n^{\alpha}}} f(n) = x/L(x)^{1+\alpha+o(1)}.$$

We deduce, on the abc-conjecture, a nontrivial upper bound on how often values of certain polynomials appear in the range of Euler's φ-function.

1. Introduction.

By a *multiplicative partition* (or *unordered factorization*) of n, we mean a way of decomposing n as a product of integers larger than 1, where two decompositions are considered the same if they differ only in the order of the factors. Let $f(n)$ denote the number of multiplicative partitions of n. For example, $f(12) = 4$, corresponding to the factorizations

$$2 \cdot 6, \quad 2 \cdot 2 \cdot 3, \quad 3 \cdot 4, \quad \text{and} \quad 12.$$

The function $f(n)$ was introduced by MacMahon in 1923 and was shortly afterwards the subject of two papers by Oppenheim [**10, 11**]. The main result of Oppenheim's first paper concerns the maximum size of $f(n)$. Let $\log_k x$ denoting the kth iterate of the natural logarithm, and put

$$L(x) = \exp\left(\log x \cdot \frac{\log_3 x}{\log_2 x}\right).$$

In [**10**], Oppenheim claims to prove that $f(n) \le n/L(n)^{2+o(1)}$, as $n \to \infty$, and that this is optimal: there is an infinite, increasing sequence of positive integers n along which $f(n) = n/L(n)^{2+o(1)}$. However, in 1983, Canfield, Erdős, and Pomerance [**2**] disproved

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Oppenhein's "theorem", showing that the true maximal order is $n/L(n)^{1+o(1)}$; more precisely,

$$\max_{n \le x} f(n) = x/L(x)^{1+o(1)}, \tag{1}$$

as $x \to \infty$.

Oppenheim's "proof" that $f(n) \le n/L(n)^{2+o(1)}$ rests on a mistaken assertion concerning the maximal order of the k-fold divisor function $d_k(n)$. Specifically, Oppenheim claims that

$$d_k(n) < k^{\log n/\log_2 n + \log n/(\log_2 n)^2 + O(\log n/(\log_2 n)^3)} \tag{2}$$

for all large n and all k. Now (2) is true when $k = 2$ (a result of Ramanujan [**16**]), and in fact true for each fixed k (see [**4**] for sharper results), but it is *not* true uniformly in k, and this invalidates his argument. Upper bounds for $d_k(n)$ which are uniform in k were eventually supplied by Usol′tsev [**20**] and Norton [**9**], and using Norton's work one can prove that $f(n) \le n/L(n)^{1+o(1)}$ along the lines envisioned by Oppenheim. (The proof in [**2**] is different, not relying on bounds for $d_k(n)$.)

It is worth observing that (2) does hold uniformly in k under the restriction that n is squarefree. In that case, $d_k(n) = k^{\omega(n)}$, and it is known that

$$\omega(n) \le \frac{\log n}{\log_2 n} + \frac{\log n}{(\log_2 n)^2} + O\left(\frac{\log n}{(\log_2 n)^3}\right). \tag{3}$$

(The estimate (3) follows from the prime number theorem with error term. See [**17**, Théorème 16] for an explicit determination of the O-constant.) Following Oppenheim's arguments leads one to a correct proof that

$$\max_{\substack{n \le x \\ n \text{ squarefree}}} f(n) = x/L(x)^{2+o(1)}, \tag{4}$$

as $x \to \infty$. This asymptotic formula can also be obtained in other ways. For instance, one can note that when n is squarefree with k prime factors, $f(n)$ is the number of set partitions of a k-element set, i.e., the kth *Bell number*. A sharp form of the prime number theorem, together with known bounds on Bell numbers (as in Lemma 6 below), easily yields (4).

The main result of this note is the following "convex combination" of the estimates (1) and (4). As usual, $\mathrm{rad}(n)$ denotes the radical of n, i.e., its largest squarefree divisor.

Theorem 1. *Fix $\alpha \in [0, 1]$. As $x \to \infty$,*

$$\max_{\substack{n \le x \\ \mathrm{rad}(n) \ge n^\alpha}} f(n) = x/L(x)^{1+\alpha+o(1)}.$$

Theorem 1 emerged during the author's investigations into the value-distribution of Euler's φ-function. Let $F(n) = \#\varphi^{-1}(n)$ denote the number of φ-preimages of n. It was shown by Pomerance [**13**] that

$$\max_{n \le x} F(n) \le x/L(x)^{1+o(1)},$$

and that equality holds if one assumes plausible conjectures on the distribution of shifted primes without large prime factors (see also [**15**]). Under the same conjectures, arguments of Banks, Friedlander, Pomerance, and Shparlinski [**1**] establish that for each fixed positive integer k,

$$\max_{n^k \le x} F(n^k) = x/L(x)^{1+o(1)}.$$

(A proof is carried out explicitly in [**12**]; be careful to note the Remark at the top of that article's page 4.) Thus, for the polynomial $P(T) = T^k$, there are values of $P(T)$ that are essentially "as popular as possible" in terms of the multiplicity with which they appear in the range of φ. Under the abc conjecture, we deduce from Theorem 1 a contrasting result when $P(T)$ has at least two distinct roots.

THEOREM 2 (conditional on the abc conjecture). *Let $P(T) \in \mathbb{Z}[T]$ be a nonzero polynomial with at least two distinct complex roots. There is a constant $c_P > 0$ such that, as $x \to \infty$,*

$$\max_{n:\ 0<|P(n)|\le x} F(|P(n)|) \le x/L(x)^{1+c_P+o(1)}.$$

2. 'Radically' refining (1) and (4): Proof of Theorem 1

Since (1) and (4) cover the cases $\alpha = 0$ and $\alpha = 1$ of Theorem 1, we will assume that $0 < \alpha < 1$. We treat the upper bound half of Theorem 1 first. The following estimate of Oppenheim plays a central role.

LEMMA 3 (see eq. (1.52) in [**10**]). *There are constants $C_1, C_2 > 0$ such that, for all positive integers $n \ge 16$ $(> e^e)$,*

$$f(n) \le C_1 \log n \cdot \max_{1 \le k \le \frac{\log n}{\log 2}} \frac{d_k(n)}{k!}(C_2 \log_3 n)^k. \tag{5}$$

The next two lemmas are due to Norton.

LEMMA 4 (see eq. (1.32) in [**9**]). *For all integers $n \ge 16$ and all integers k with $2 \le k \le \frac{2\log n}{\log_2 n}$, we have*

$$\log d_k(n) \le \log k \frac{\log n}{\log_2 n}\left(1 + \frac{\log_3 n}{\log_2 n} + O\left(\frac{1}{\log_2 n} + \frac{k \log_3 n}{\log n}\right)\right).$$

LEMMA 5 (see eq. (1.34) in [**9**]). *For all $k \ge 2$ and all positive integers n,*

$$d_k(n) < n^2 e^k. \tag{6}$$

REMARK. While Norton only claims (6) when $k \ge \log n$, it is clear from eq. (5.6) in [**9**] that this result holds for all $k \ge 2$.

PROOF OF THE UPPER BOUND IN THEOREM 1. It suffices to show that for each $\epsilon \in (0, 1/2)$, the maximum appearing in Theorem 1 is $O_\epsilon(x/L(x)^{1+\alpha-2\epsilon})$ for all $x > x_0(\epsilon)$. Of course, $f(n) \le x/L(x)^{1+\alpha}$ when n is bounded and $x \to \infty$, so we can and will assume when convenient that n is sufficiently large.

If the maximum in (5) occurs at k where $k \ge 2\log n/\log_2 n$, then (keeping in mind (6))

$$\begin{aligned} f(n) \ll \log n \cdot \frac{d_k(n)}{k!}(C_2 \log_3 n)^k &\le \log n \cdot \frac{n^2 \cdot e^k}{k!}(C_2 \log_3 n)^k \\ &\le n^2 \log n \cdot (C_2 e^2 \log_3 n/k)^k, \end{aligned}$$

where we used in the last step the elementary inequality $k! \geq (k/e)^k$. Our lower bound on k implies that the last displayed quantity is of size $n^{o(1)}$, as $n \to \infty$, which is smaller than $x/L(x)^2$ for large x. So the upper bound of the theorem holds in this case, with much room to spare.

Now suppose the maximum occurs at a value k with

$$1.1 \log_3 n \frac{\log n}{(\log_2 n)^2} \leq k < 2 \log n / \log_2 n.$$

Write $k = Z \log n/(\log_2 n)^2$, so that

$$1.1 \log_3 n \leq Z < 2 \log_2 n.$$

Then

$$\log k \frac{\log n}{\log_2 n} = \log n - 2 \log L(n) + O\left(\log Z \frac{\log n}{\log_2 n}\right),$$

$$\log k \frac{\log n}{\log_2 n} \cdot \frac{\log_3 n}{\log_2 n} = (1 + o(1)) \log L(n),$$

and

$$\log k \frac{\log n}{\log_2 n} \cdot \left(\frac{1}{\log_2 n} + \frac{k \log_3 n}{\log n}\right) = o(\log L(n)) + O\left(\frac{Z}{\log_2 n} \log L(n)\right).$$

(Here and elsewhere in this paragraph, the limit implicit in the $o(\cdot)$ terms is as $n \to \infty$.) Collecting these estimates and appealing to Lemma 4 reveals that

$$\log d_k(n) \leq \log n - (1 + o(1)) \log L(n) + O\left(\frac{Z}{\log_2 n} \log L(n)\right) + O\left(\log Z \frac{\log n}{\log_2 n}\right).$$

Also,

$$\begin{aligned}\log k! &= k \log k + o(\log L(n)) \\ &\geq k(\log_2 n - 2 \log_3 n) + o(\log L(n)) \\ &= Z \frac{\log n}{\log_2 n}(1 - O(\log_3 n/\log_2 n)) + o(\log L(n)).\end{aligned}$$

Moreover, $\log((C_2 \log_3 n)^k) = o(\log L(n))$. So from (5), as $n \to \infty$,

$$\begin{aligned}\log f(n) &\leq O(1) + \log_2 n + \log d_k(n) - \log k! + \log((C_2 \log_3 n)^k) \\ &\leq \log n - (1 + o(1)) \log L(n) - Z \frac{\log n}{\log_2 n}(1 + o(1)).\end{aligned}$$

Inserting our lower bound on Z and exponentiating, we find that for large n,

$$f(n) \leq n/L(n)^2 \leq x/L(x)^2.$$

Thus, the upper bound in the theorem holds in this case as well.

We may therefore suppose the maximum occurs at $k < 1.1 \frac{\log n \log_3 n}{(\log_2 n)^2}$. Write $n = AB$, where $A = \mathrm{rad}(n)$. Recall that d_k is a submultiplicative function, meaning that $d_k(ab) \leq d_k(a) d_k(b)$ for every pair of positive integers a, b (see for instance [**18**]). Thus,

$$f(n) = f(AB) \ll \log n \cdot d_k(A) \frac{d_k(B)}{k!} \cdot (C_2 \log_3 n)^k.$$

Applying (3) with n replaced by A shows that

$$\log d_k(A) = \omega(A)\log k \le \left(\frac{\log A}{\log_2 A} + O(\log A/(\log_2 A)^2)\right)\log k. \tag{7}$$

Recall that $A \ge n^\alpha$. Thus, assuming (as we may) that n is large, we have that

$$\begin{aligned}\log k &\le \log_2 n - 2\log_3 n + O(\log_4 n)\\ &\le \log_2 A - 2\log_3 A + O(\log_4 A),\end{aligned}$$

so that (7) yields $\log d_k(A) \le \log A - (2+o(1))\log L(A)$, as $n \to \infty$. So for large n,

$$d_k(A) < A/L(A)^{2-\epsilon}.$$

Also, $\log n \cdot (C_2 \log_3 n)^k \ll L(x)^\epsilon$, and a moment's thought reveals that $d_k(B)/k! \le f(B)$. Hence, if n is large, then

$$f(n) \ll \frac{A}{L(A)^{2-\epsilon}} f(B) L(x)^\epsilon.$$

If $B < 16$, then this upper bound is $O(x/L(x)^{2-2\epsilon})$, completing the proof of the theorem. If $B \ge 16$, we have from (1) that $f(B) \ll B/L(B)^{1-\epsilon}$, making

$$f(n) \ll \frac{n}{L(A)^{2-\epsilon}L(B)^{1-\epsilon}} L(x)^\epsilon.$$

To finish things off, notice that

$$\begin{aligned}\log(L(A)^{2-\epsilon}L(B)^{1-\epsilon}) &= (2-\epsilon)\frac{\log A \log_3 A}{\log_2 A} + (1-\epsilon)\frac{\log B \log_3 B}{\log_2 B}\\ &\ge (2-\epsilon)\frac{\log A \log_3 n}{\log_2 n} + (1-\epsilon)\frac{\log B \log_3 n}{\log_2 n}\\ &= \frac{\log A \log_3 n}{\log_2 n} + (1-\epsilon)\log L(n) \ge (1+\alpha-\epsilon)\log L(n).\end{aligned}$$

Hence,

$$f(n) \ll \frac{n}{L(n)^{1+\alpha-\epsilon}} L(x)^\epsilon \ll \frac{x}{L(x)^{1+\alpha-2\epsilon}},$$

as desired. □

The lower bound half of Theorem 1 is easier. We use the following asymptotic estimate for Bell numbers, which is a weak form of a result proved in [**3**, Chapter 6].

Lemma 6. *The kth Bell number B_k satisfies, for $k \ge 3$,*

$$\log B_k = k\log k - k\log\log k + O(k).$$

Proof of the lower bound in Theorem 1. It is enough to show that for all $\epsilon \in (0, \frac{1}{2}(1-\alpha))$, the maximum indicated in Theorem 1 is at least $x/L(x)^{1+\alpha+4\epsilon}$ once $x > x_0(\epsilon)$. Let B be a positive integer in $[1, x^{1-\alpha-\epsilon}]$ for which $f(B)$ is as as large as possible; from (1), we know that for large x,

$$f(B) > x^{1-\alpha-\epsilon}/L(x^{1-\alpha-\epsilon})^{1+\frac{1}{2}\epsilon} \ge B/L(B)^{1+\frac{1}{2}\epsilon}. \tag{8}$$

Since the value of $f(B)$ depends only on the array of exponents in the prime factorization of B, and not on the primes themselves, we may assume that the primes dividing B are precisely the primes not exceeding its largest prime factor q. Since $B \le x^{1-\alpha-\epsilon}$, we have by the prime number theorem that $q < \log x$ (for large x). Let A be the product of the consecutive primes exceeding $\log x$, with the product extending as far as possible with $A \le x/B$. Then A and B are relatively prime, and so (concatenating factorizations) we see that

$$f(AB) \ge f(A)f(B). \tag{9}$$

The proof will be completed by showing that $\mathrm{rad}(AB) \ge x^{\alpha}$, and that $f(A)f(B) \ge x/L(x)^{1+\alpha+4\epsilon}$.

To get started, we observe that $(\log x)^{\omega(A)} \le A \le x/B$, so that

$$\omega(A) \le \log{(x/B)}/\log_2(x).$$

For large x, this implies that the first $\omega(A)+1$ primes exceeding $\log x$ all belong to the interval $(\log x, 3\log x]$. Now the choice of A implies that

$$\mathrm{rad}(AB) \ge \mathrm{rad}(A) = A \ge \frac{x}{3B\log x} \ge x^{\alpha}.$$

Moreover, since $(3\log x)^{\omega(A)} \ge A \ge x/(3B\log x)$, we have

$$\begin{aligned} \omega(A) &\ge \log(x/(3B\log x))/\log(3\log x) \\ &= \frac{\log(x/B)}{\log_2 x}\left(1+O(1/\log_2 x)\right). \end{aligned}$$

Combining this with our earlier upper bound for $\omega(A)$, we see that

$$\omega(A) = \frac{\log(x/B)}{\log_2 x}\left(1+O(1/\log_2 x)\right).$$

Hence

$$\log\omega(A) = \log_2 x - \log_3 x + O(1), \quad \log_2\omega(A) = \log_3 x + O(\log_3 x/\log_2 x).$$

Now a straightforward calculation using Lemma 6 reveals that

$$\log f(A) = \log B_{\omega(A)} = \log(x/B) - 2\log(x/B)\frac{\log_3 x}{\log_2 x} + o(\log L(x)),$$

as $x \to \infty$. Recalling (8) and (9), and observing that (8) implies that $B > x^{1-\alpha-2\epsilon}$, we find that

$$\begin{aligned} \log f(AB) &\ge \log f(A) + \log f(B) \\ &\ge \log x - 2\log(x/B)\frac{\log_3 x}{\log_2 x} - \left(1+\frac{1}{2}\epsilon\right)\log B\frac{\log_3 B}{\log_2 B} + o(\log L(x)) \\ &\ge \log x - 2\log(x/B)\frac{\log_3 x}{\log_2 x} - (1+\epsilon)\log B\frac{\log_3 x}{\log_2 x} + o(\log L(x)) \\ &= \log x - (1+o(1))\log L(x) - \log(x/B)\frac{\log_3 x}{\log_2 x} - \epsilon\log B\frac{\log_3 x}{\log_2 x}. \end{aligned}$$

Since $\frac{x}{B} \le x^{\alpha+2\epsilon}$ and $B \le x$, we deduce that

$$\log f(AB) \ge \log x - (1+\alpha+3\epsilon+o(1))\log L(x),$$

and so for large x, we have $f(AB) \ge x/L(x)^{1+\alpha+4\epsilon}$, as desired. $\square$

3. (Un)popular polynomial values: Proof of Theorem 2

We use the following consequence of the abc conjecture, due to Langevin [**8**] (see also Granville [**7**]).

PROPOSITION 7 (conditional on abc). *Fix a polynomial $P(T) \in \mathbb{Z}[T]$ with degree $d \geq 2$ and no repeated roots. Then*

$$\mathrm{rad}(P(n)) \geq |n|^{d-1-o(1)},$$

for integers n with $|n| \to \infty$.

We also need the following lemma comparing $F(n)$ and $f(n)$. A similar result, for the Dedekind ψ-function in place of φ, was given by Pomerance [**14**].

LEMMA 8. *For every positive integer n, we have $F(n) \leq 4f(n)$.*

PROOF. By an *extended factorization* of n, we mean a multiplicative partition of n, but with 1 now allowed to appear as a factor at most once. Clearly, the number of extended factorizations of n is exactly $2f(n)$.

Suppose that $\varphi(m) = n$, and write $m = p_1^{e_1} \cdots p_k^{e_k}$, where the p_i are distinct primes and the e_i are positive integers; then

$$n = \prod_{i=1}^{k} \overbrace{p_i \cdots p_i \cdots p_i}^{e_i - 1 \text{ times}} \cdot (p_i - 1).$$

We view the right-hand side as describing an extended factorization of n into $\sum_{i=1}^{k}((e_i - 1) + 1) = \Omega(m)$ parts. It suffices to show that each such extended factorization of $\varphi(m)$ corresponds to at most two different preimages m.

Starting from the (extended) factorization of $\varphi(m)$, one might attempt to recover m as follows: Such a factorization contains a unique smallest term $p_1 - 1$, where p_1 is the smallest prime factor of m. We read off the multiplicity of p_1 in m as 1 + the multiplicity of p_1 in our factorization. We then remove $p_1 - 1$ and the copies of p_1 from our factorization and start over to determine the next largest prime factor of m and the multiplicity with which it appears. We continue in this way until the entire factorization of $\varphi(m)$ is exhausted.

However, this procedure can fail if $p_1 = 2$, since the number of 2's appearing in the factorization of $\varphi(m)$ depends not only on the power of 2 in m but also on whether or not $3 \mid m$. We work around this as follows:

- If $p_1 = 2$, and the factorization of $\varphi(m)$ contains 3 as a term, we know $3 \mid m$, which is enough to resolve all ambiguity: If 2 appears in the given factorization of $\varphi(m)$ k times, then $2^k \parallel m$. Having figured out the power of 2 dividing m, we remove the factor 1 and $k - 1$ factors of 2 from the factorization of $\varphi(m)$ and proceed to determine the remaining components of m by the procedure of the last paragraph.
- Suppose $p_1 = 2$ and 3 does not appear in the factorization of $\varphi(m)$. If 2 appears in the factorization k times, then either $2^{k+1} \parallel m$ or $2^k \cdot 3 \parallel m$. In either case, we may remove 1 and all factors of 2 from the factorization of $\varphi(m)$ and continue with the algorithm above to determine the remaining components of m.

In either case, the factorization of $\varphi(m)$ determines m in at most two ways, finishing the proof. $\square$

Proof of Theorem 2. Write $P(T) = \pm \prod_i P_i(T)^{e_i}$, where the $P_i(T)$ are distinct nonconstant irreducibles in $\mathbb{Z}[T]$, each with positive leading coefficient. Let $Q(T) = \prod_i P_i(T)$. Then $Q(T)$ has distinct roots, and $d := \deg Q(T) \geq 2$. By Proposition 7, as $|n| \to \infty$,

$$\mathrm{rad}(P(n)) \geq \mathrm{rad}(Q(n)) \geq |n|^{d-1-o(1)} \geq |P(n)|^{1-\frac{1}{d}-o(1)}.$$

It follows from Lemma 8 and Theorem 1 that

$$F(P(n)) \leq 4f(P(n)) \leq |P(n)|/L(|P(n)|)^{2-\frac{1}{d}+o(1)}.$$

Theorem 2 follows with $c_P = 1 - \frac{1}{d}$. $\square$

Probably the conclusion of Theorem 2 is still quite far from the truth. For each fixed $\alpha \in [0, 1]$,

$$\#\{n \leq x : F(n) \geq x^{\alpha}\} \leq x^{1-\alpha+o(1)}, \tag{10}$$

as $x \to \infty$. This follows from the easy estimate $\#\varphi^{-1}([1, x]) \ll x \log_2 x$. (In fact, $x \log_2 x$ can be improved to x [**5**].) Now a naive probabilistic argument suggests that if $P(T)$ is a polynomial of degree d, then

$$\max_{n:\ 0<|P(n)|\leq x} F(|P(n)|) \leq x^{\frac{1}{d}+o(1)}, \tag{11}$$

as $x \to \infty$. This conclusion should be taken with a grain of salt; for the polynomials $P(T) = T^k$, this maximum can be shown rigorously to be at least $x^{0.7038}$ (and as pointed out in the introduction, we expect it to be $x^{1-o(1)}$). (See [**6**], which develops arguments of [**1**].) But it may be that (11) holds generically, perhaps whenever $P(T)$ has distinct roots.

Remark. It would also be sensible to study $f(|P(n)|)$ rather than $F(|P(n)|)$. Clearly, Theorem 2 remains valid in this context. Moreover, one can show rigorously that $\max_{n^k \leq x} f(n^k) = x/L(x)^{1+o(1)}$, as $x \to \infty$.[1] Since $\sum_{n \leq x} f(n) \leq x \exp(O(\sqrt{\log x}))$ (see [**11**] or [**19**] for an asymptotic formula), one has the analogue of (10), and our probabilistic heuristic suggests that the analogue of (11) holds for a generic choice of $P(T)$.

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[1] In view of (1), only the implicit lower bound needs discussion. One modifies the proof of Theorem 1 in [**12**] as follows: Rather than choose t distinct primes $p \leq X$ with $p-1$ y-smooth, choose t y-smooth integers $m_1 < \cdots < m_t \leq X$. Each choice of $m_1, \ldots, m_t$ corresponds to a factorization of $N := m_1 \cdots m_t$, where $N \leq x$. The argument for Theorem 1 in [**12**] shows that there are at least $x/L(x)^{1+o(1)}$ choices of the m_i for which N is a kth power. But N is y-smooth, and there are only $L(x)^{o(1)}$ y-smooth integers in $[1, x]$. Thus, $f(N) \geq x/L(x)^{1+o(1)}$ for some N.

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